λ

Today

- λ-calculus
- β-reduction
- Church encoding

Edward Sapir

Benjamin Whorf

Sapir-Whorf Hypothesis

Edward Sapir

Language limits thought.

Benjamin Whorf

```
function f(n) {
  if (n == 0)
    return 1 ;
  else
    return n * f(n-1) ;
}
```

```
function f(n) {
  var a = 1 ;
  while (n > 0) {
    a = a * n ;
    n-- ;
  }
  return a ;
}
```

```
function f(a,n) {
  if (n == 0)
    return a ;
  else
    return f(a*n,n-1) ;
}
```

```
function f(n) {
 return (n <= 0) ? 1 : n * f(n-1) ;
}
```

```
(function (h)
  function (n) (n <= 1) ?
     1 : n*(h(h))(n-1))
 (function (h)
   function (n) (n <= 1) ?
      1 : n*(h(h))(n-1))(5)
```

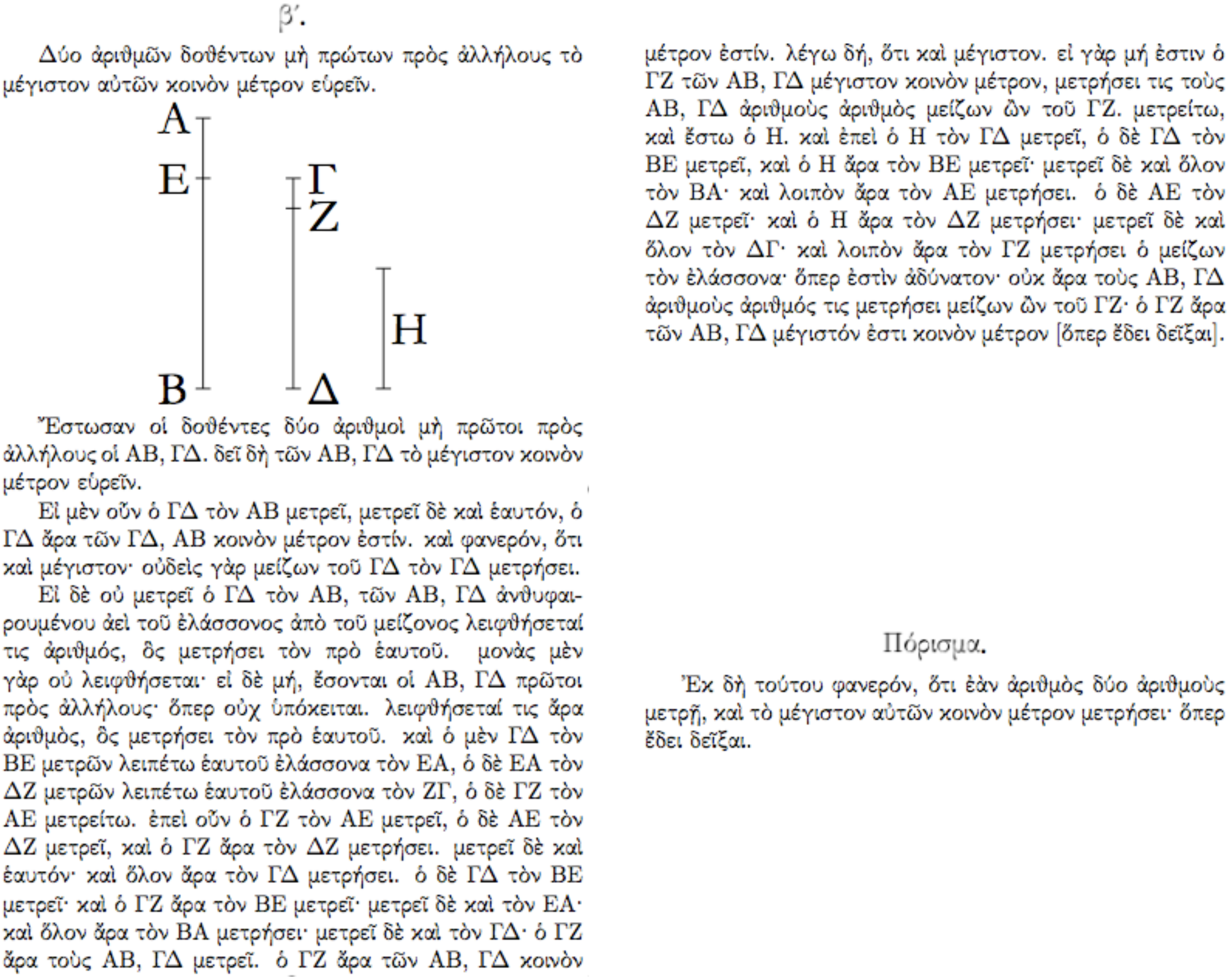
Origins of notation

Euclid's algorithm (300 BC)

- Greatest common divisor
- GCD(24,18) = 6
- GCD(4,6) = 2

(Probably) Euclid

300 BC



β′.

Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ AB, ΓΔ. δεῖ δὴ τῶν AB, ΓΔ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰ μὲν οὖν ὁ ΓΔ τὸν AB μετρεῖ, μετρεῖ δὲ καὶ ἑαυτόν, ὁ ΓΔ ἄρα τῶν ΓΔ, AB κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεὶς γὰρ μείζων τοῦ ΓΔ τὸν ΓΔ μετρήσει.

Εἰ δὲ οὐ μετρεῖ ὁ ΓΔ τὸν AB, τῶν AB, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος λειφθήσεταί τις ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. μονὰς μὲν γὰρ οὐ λειφθήσεται· εἰ δὲ μή, ἔσονται οἱ AB, ΓΔ πρῶτοι πρὸς ἀλλήλους· ὅπερ οὐχ ὑπόκειται. λειφθήσεταί τις ἄρα ἀριθμὸς, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. καὶ ὁ μὲν ΓΔ τὸν BE μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν EA, ὁ δὲ EA τὸν ΔZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ZΓ, ὁ δὲ ΓZ τὸν AE μετρείτω. ἐπεὶ οὖν ὁ ΓZ τὸν AE μετρεῖ, ὁ δὲ AE τὸν ΔZ μετρεῖ, καὶ ὁ ΓZ ἄρα τὸν ΔZ μετρήσει. μετρεῖ δὲ καὶ ἑαυτόν· καὶ ὅλον ἄρα τὸν ΓΔ μετρήσει. ὁ δὲ ΓΔ τὸν BE μετρεῖ· καὶ ὁ ΓZ ἄρα τὸν BE μετρεῖ· μετρεῖ δὲ καὶ τὸν EA· καὶ ὅλον ἄρα τὸν BA μετρήσει· μετρεῖ δὲ καὶ τὸν ΓΔ· ὁ ΓZ ἄρα τοὺς AB, ΓΔ μετρεῖ. ὁ ΓZ ἄρα τῶν AB, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ ΓZ τῶν AB, ΓΔ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς AB, ΓΔ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ ΓZ. μετρείτω, καὶ ἔστω ὁ H. καὶ ἐπεὶ ὁ H τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν BE μετρεῖ, καὶ ὁ H ἄρα τὸν BE μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν BA· καὶ λοιπὸν ἄρα τὸν AE μετρήσει. ὁ δὲ AE τὸν ΔZ μετρεῖ· καὶ ὁ H ἄρα τὸν ΔZ μετρήσει· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓZ μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα τοὺς AB, ΓΔ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ ΓZ· ὁ ΓZ ἄρα τῶν AB, ΓΔ μέγιστόν ἐστι κοινὸν μέτρον [ὅπερ ἔδει δεῖξαι].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει· ὅπερ ἔδει δεῖξαι.

2008 AD

```
gcd(a, b) = (b == 0) ? a : gcd(b, a mod b)
```

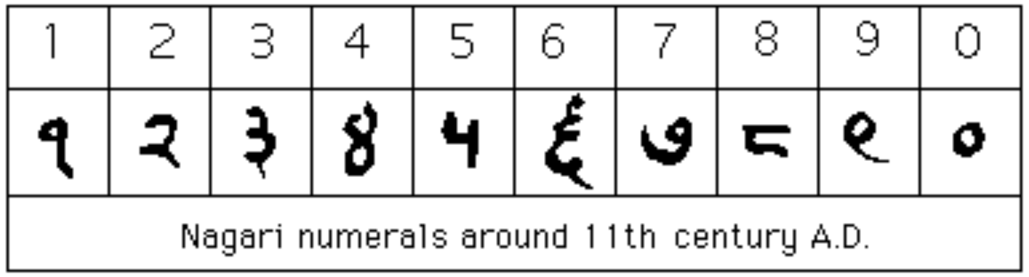
Limits on the Greeks

- No notation for zero.
- No variables for unknowns.
- No symbols for operations.
- Long division required Ph.D.
- Irrational numbers punished by death.

Example

- The number such that four of its roots is equal to its three of its square.
- $4x = 3x^2$.

Indian numerals (596)



1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०
Nagari numerals around 11th century A.D.									

- Notation for zero.
- Decimal numerals.
- Calculation easier.

Brahmagupta

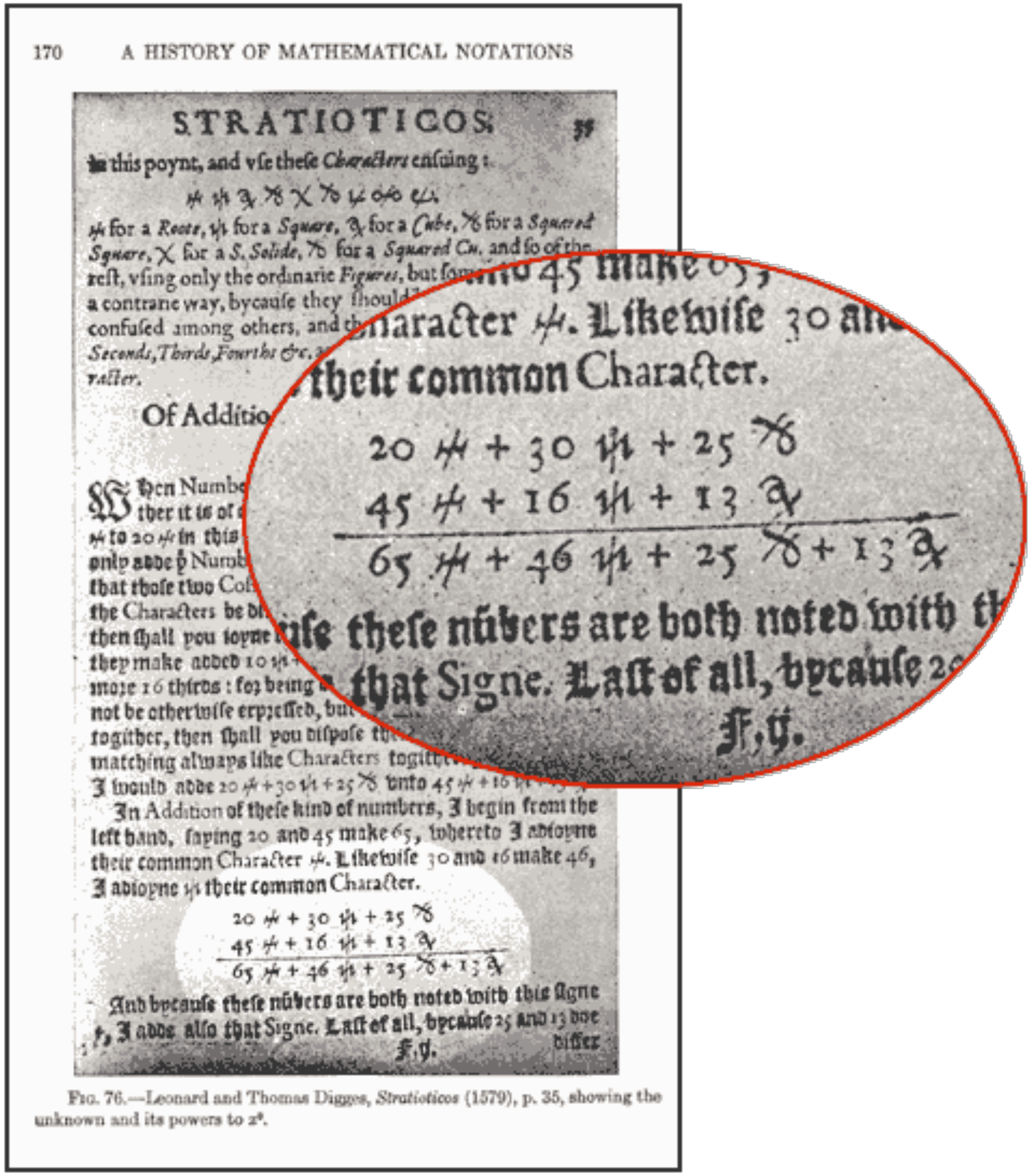
et

t

t

+

Variables (1570s)



170 A HISTORY OF MATHEMATICAL NOTATIONS

STRATIOTICOS. 35

In this poynt, and vſe theſe Characters enſuing:

for a Roote, for a Square, for a Cube, for a Squared Square, for a S. Solide, for a Squared Cu. and ſo of the reſt, vſing only the ordinarie Figures, but

a contrarie way, bycauſe they ſhould

confuſed among others, and th

Seconds, Thirds, Fourths &c.

racter.

Of Additio

In Addition of theſe kind of numbers, I begin from the left hand, ſaying 20 and 45 make 65, whereto I adioyne their common Character. Likewiſe 30 and 16 make 46, I adioyne their common Character.

And bycauſe theſe nūbers are both noted with this ſigne, I adde alſo that Signe. Laſt of all, bycauſe 25 and 13 doe differ

F.ij.

Fig. 76.—Leonard and Thomas Digges, *Stratioticos* (1579), p. 35, showing the unknown and its powers to x^8.

François Viète

Example

- $3 + x$ is equal to 10 times x^2.
- $3 + x = 10x^2$

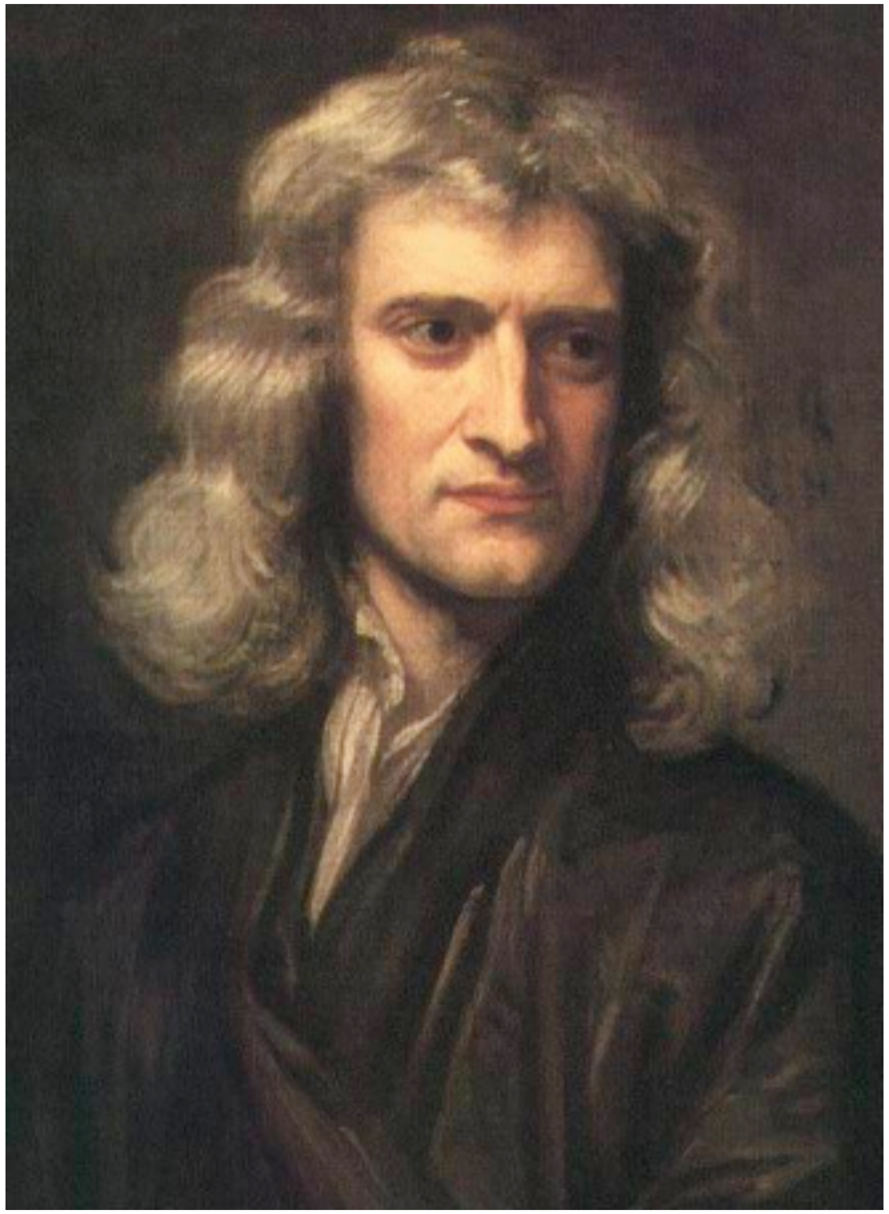
Operations (Early 1600s)

- Letters for variables.
- Symbols for operations.
- Led to slide rule.

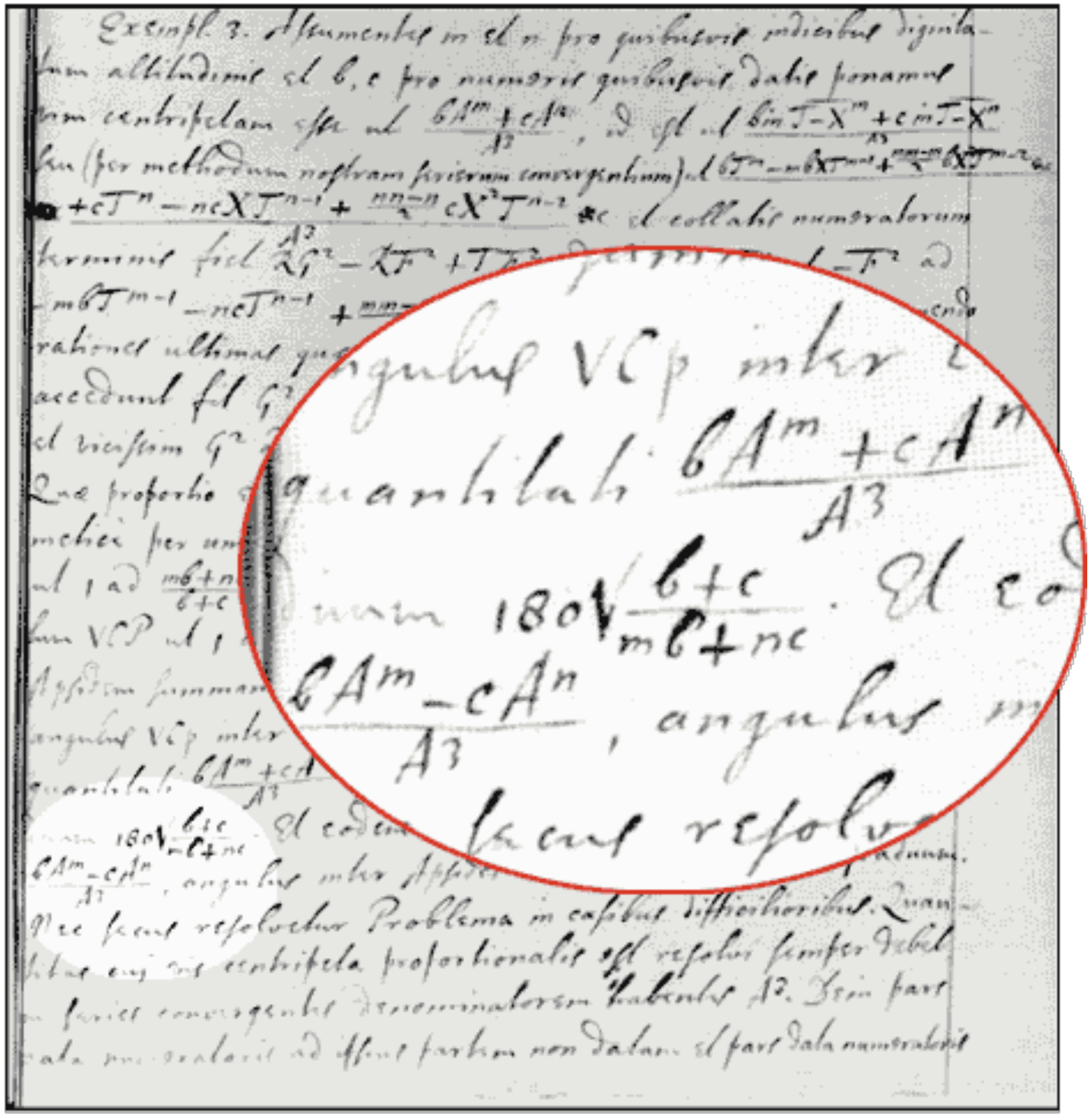
SYMBOLS	MEANINGS OF SYMBOLS
	Commens. potentia
	Incommens. potentia
	Rationale
	Irrationale
	Medium
	Line, cut extr. and mean ratio
	Major ejus portio
	Minor ejus portio
sim	Simile
	Proxime majus
	Proxime minus
	Aequale vel minus
	Aequale vel majus
▯	*Rectangulum*
□	*Quadratum*
△	*Triangulum*
	Latus, radix
	Media proportion
∽	*Differentia*[24]
‖	Parallel
log	Logarithm
log:Q:	Log. of square
S	Sine[25]
t	Tangent
se	Secant
sv	*Sinus versus*
s ver	*Sinus versus*[26]
sin : com	Sine complement
s co	Cosine
t co	Cotangent
se co	Cosecant
sin	Sine
tan	Tangent
sec	Secant
sec:parall	Sum of secants

William Oughtred

Calculus (Late 1600s)



Isaac Newton



Principia (1687)

Calculus (Late 1600s)

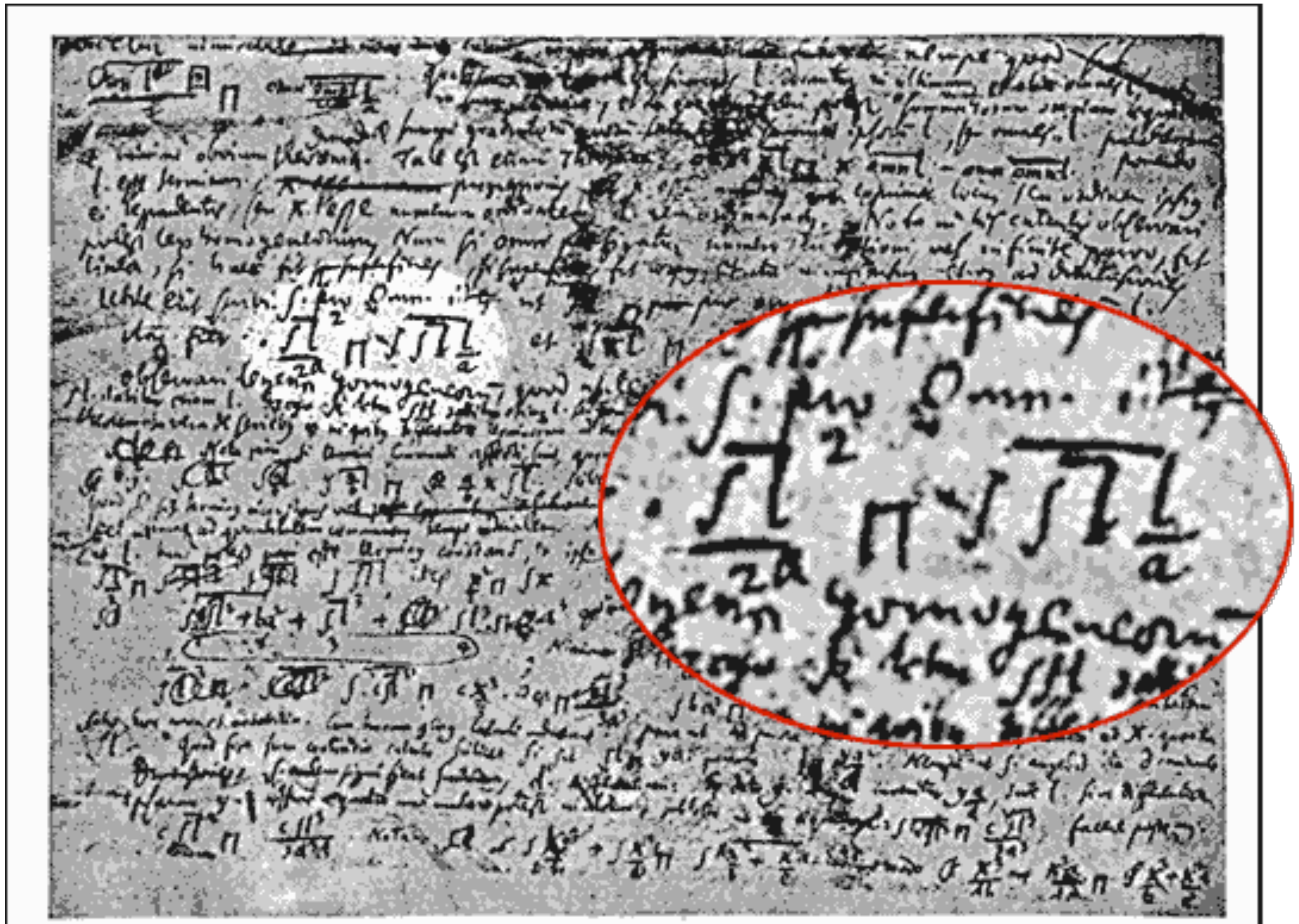
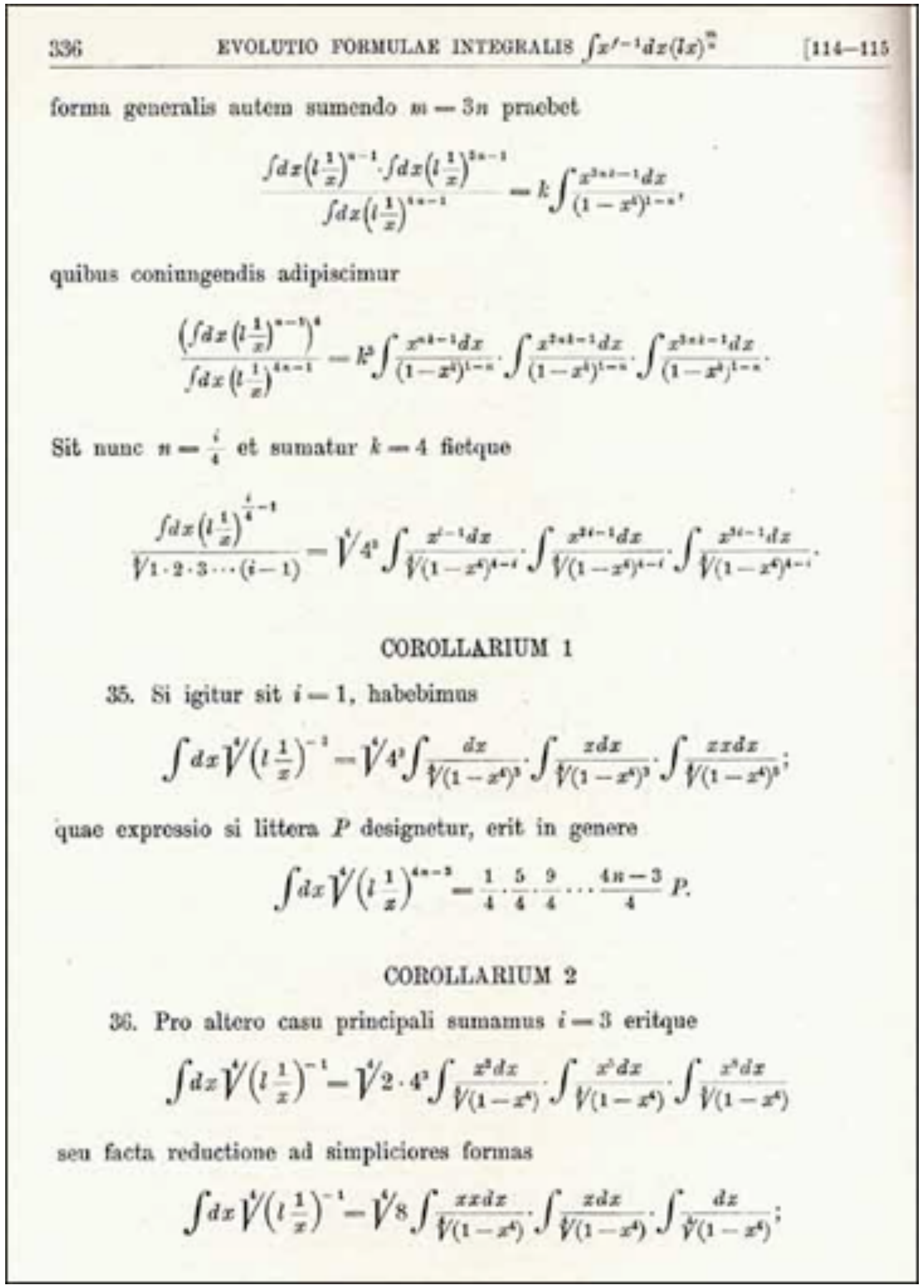


FIG. 124.—Facsimile of manuscript of Leibniz, dated Oct. 29, 1675, in which his sign of integration first appears. (Taken from C. I. Gerhardt's *Briefwechsel von G. W. Leibniz mit Mathematikern* [1899].)

Gottfried Leibniz

Euler (1700s)



336 EVOLUTIO FORMULAE INTEGRALIS $\int x^{f-1}dx(lx)^{\frac{m}{n}}$ [114–115

forma generalis autem sumendo $m = 3n$ praebet

$$\frac{\int dx\left(l\frac{1}{x}\right)^{n-1}\cdot\int dx\left(l\frac{1}{x}\right)^{3n-1}}{\int dx\left(l\frac{1}{x}\right)^{4n-1}} = k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}},$$

quibus coniungendis adipiscimur

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^4}{\int dx\left(l\frac{1}{x}\right)^{4n-1}} = k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}.$$

Sit nunc $n = \frac{i}{4}$ et sumatur $k = 4$ fietque

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1\cdot2\cdot3\cdots(i-1)}} = \sqrt[4]{4^3}\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}.$$

COROLLARIUM 1

35. Si igitur sit $i = 1$, habebimus

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-3}} = \sqrt[4]{4^3}\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}};$$

quae expressio si littera P designetur, erit in genere

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{4n-3}} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\frac{4n-3}{4}P.$$

COROLLARIUM 2

36. Pro altero casu principali sumamus $i = 3$ eritque

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-1}} = \sqrt[4]{2\cdot4^3}\int\frac{x^2dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^5dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^8dx}{\sqrt[4]{(1-x^4)}}$$

seu facta reductione ad simpliciores formas

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-1}} = \sqrt[4]{8}\int\frac{xxdx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{xdx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{dx}{\sqrt[4]{(1-x^4)}};$$

The end of the reign of numbers

Functions

A **function** transforms an input into an output.

$$f(x) = x^2 + 3$$

input: x	output: $f(x)$
0	3
1	4
2	7

Names for functions

$$f(x)$$

Names for functions

f

Names for functions

$$f^{(3)}$$

Sets and Logic (1800s)

- Sets became objects.
- Logic became math.
- Math began unifying.

Giuseppe Peano

Frege’s unification

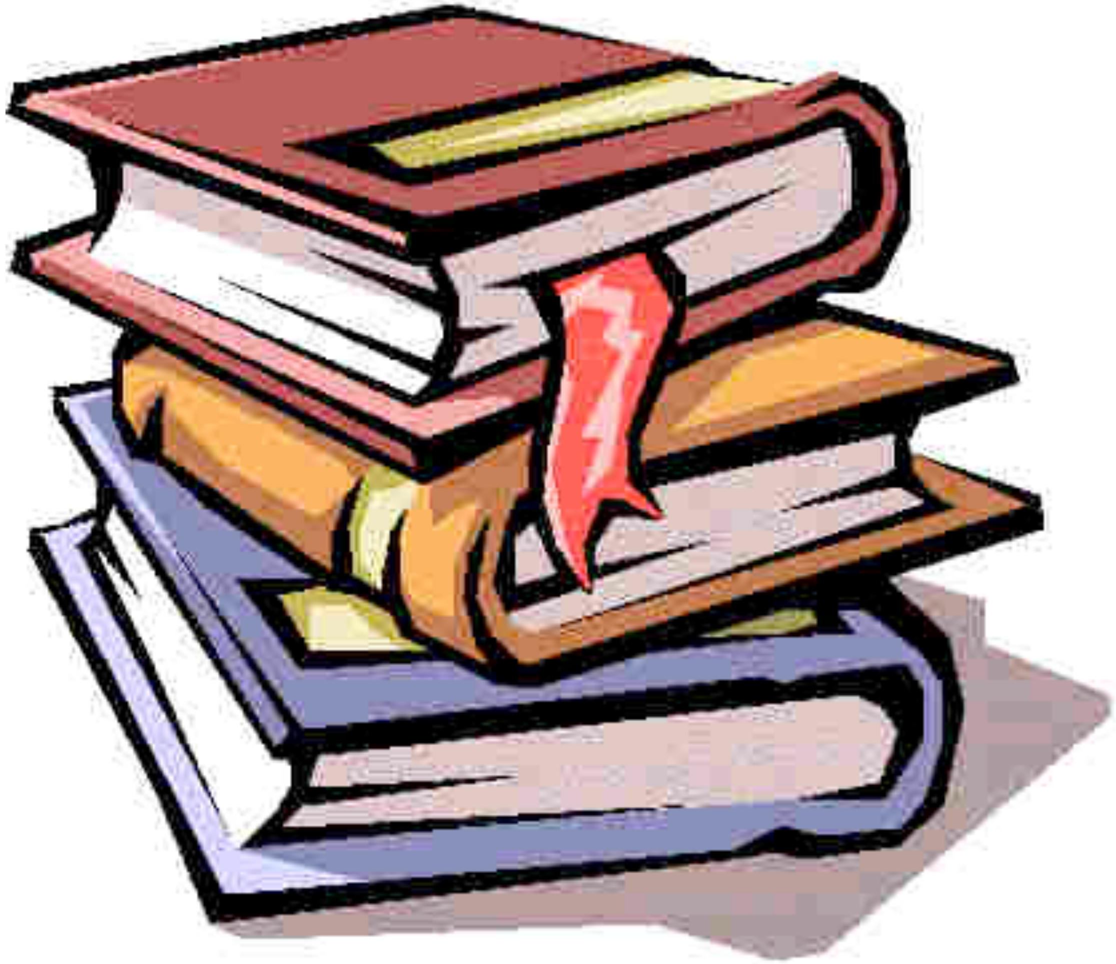
- Logic as foundation.
- Sets as atoms.
- Numbers from sets.
- Functions from sets.

Gottlob Frege

Example

- Every even integer greater than two can be written as the sum of two primes.
- $\forall n > 2 : \exists a,b : p(a) \wedge p(b) \wedge a+b = n.$

Grundgesetze der Arithmetik



- Published in 1903.
- Foundation for math.

Russell’s postcard

Dear Frege,

$$\{X \mid X \notin X\} \in \{X \mid X \notin X\}$$

Prof. Dr. Gottlob Frege
University of Göttingen
Göttingen, Germany

XOXO,
Bertrand R.

Russell’s postcard

Dear Frege,

FAIL.

Prof. Dr. Gottlob Frege
University of Göttingen
Göttingen, Germany

XOXO,
Bertrand R.

Russell's paradox

$$\{X \mid X \notin X\} \in \{X \mid X \notin X\}$$

Russell's paradox

$$\{X \mid X \notin X\} \in \{X \mid X \notin X\}$$

Does the set of all sets that do not contain themselves contain itself?

Russell's paradox

$$\{X \mid X \notin X\} \in \{X \mid X \notin X\}$$

Does the set of all sets that do not contain themselves contain itself?

The barber shaves all those that do not shave themselves.

Russell’s paradox

$$\{X \mid X \notin X\} \in \{X \mid X \notin X\}$$

Does the set of all sets that do not contain themselves contain itself?

The barber shaves all those that do not shave themselves.

But, then who shaves the barber?

Russell's solution: Orders

- Problem is self-reference.
- Example: This sentence is false.
- Solution: Order sentences.
- Must reference lower orders.
- *Seems* to avoid paradox.

Bertrand Russell

Functions as foundation?

- Notation for functions.
- $f(x) = x^2$
- $f(2) = 4$
- $f = \lambda x.x^2$
- $(\lambda x.x^2)(2) = 4$

Alonzo Church

Lambda Calculus (1920s)

Throw away everything in math, except:

x	variables
$f(e)$	function application
$\lambda x.\, e$	anonymous function

Turing machine (1936)

- Student of Church.
- Defined computability.
- Showed λ = computer.

Alan Turing

What is λ?

An anonymous function

`function (`$v_1, \ldots, v_n$`) { return` *exp* `; }`

$\texttt{function (}v_1\texttt{,...,}v_n\texttt{)}\ \mathit{exp}$

Everyone but IE!

$$\texttt{function}\ (v_1, \ldots, v_n)\ exp$$

`lambda` v_1`,...,`v_n`:` *exp*

```
new Procedure () {
 public T run (T_1 v_1, ..., T_n v_n) {
   return exp ;
 }
}
```

$$\texttt{[}\mathit{free}\texttt{](}T_1\ \ v_1\texttt{,...,}T_n\ \ v_n\texttt{)\{ return }\mathit{exp}\texttt{; \}}$$

([ ] ( ) { }) ( )

`function (`v_1`,...,`v_n`) return` *exp* `end` `lambda {|`v_1`, . . . ,`v_n`| return` exp `}`

(lambda (v_1 ... v_n) *exp*)

$$(\lambda\ (v_1\ \ldots\ v_n)\ exp)$$

λ-calculus

$$
\begin{array}{lcl}
e & ::= & v \\
  & | & \lambda v.e \\
  & | & e_1(e_2)
\end{array}
$$

$$
\begin{array}{lcl}
e & ::= & v \\
  & | & (\lambda\ (v)\ e) \\
  & | & (e_1\ e_2)
\end{array}
$$

β-reduction

$$((\lambda\ (v)\ e_1)\ e_2)$$

$$\{e_2/v\}e_1$$

```
((λ (x) x) e)
```

e

```
(function (e) e + e)(3)
```

$3 + 3$

Church encoding

Sugar λ into language

Menu

- Multiple arguments
- Void value
- Lists
- Conditionals
- Numbers
- Recursion

Multiple arguments

$$f : X \times Y \to Z$$

$$f^C : X \to Y \to Z$$

$$f^C = \lambda x.\lambda y.f(x, y)$$

Multiple arguments

`f(x,y) => ((f x) y)`

Void

$$\mathbf{void} = \lambda\_.\_$$

Church’s trick

Encode data according to how it’s used.

Conditionals

$$\mathbf{true} \equiv \lambda c.\lambda a.c(\mathbf{void})$$

$$\mathbf{false} \equiv \lambda c.\lambda a.a(\mathbf{void})$$

$$\mathbf{if}\ e_b\ \mathbf{then}\ e_t\ \mathbf{else}\ e_f \equiv e_b\ (\lambda().e_t)\ (\lambda().e_f)$$

Numerals

$$n^C = \lambda f.\lambda z.f^n(z).$$

$$\mathbf{zero} \equiv \lambda f.\lambda z.z.$$

$$e_n + 1 \equiv \lambda f.\lambda z.f(e_n\ f\ z).$$

$$e_n + e_m \equiv \lambda f.\lambda z.(e_m\ f\ (e_n\ f\ z)).$$

$$e_m \times e_n \equiv \lambda f.\lambda z.(e_m\ (e_n\ f)\ z).$$

Lists

$$\mathbf{nil} \equiv \lambda e.\lambda l.e(\mathbf{void}).$$

$$\mathbf{cons} \equiv \lambda a.\lambda b.\lambda e.\lambda l.(l\ a\ b).$$

$$\mathbf{match}\ (e) \begin{cases} \mathbf{nil} & \mapsto e_e \\ \mathbf{cons}\ a\ b & \mapsto e_l \end{cases} \quad \equiv \quad e\ (\lambda().e_e)\ (\lambda a.\lambda b.e_l).$$

$$\langle e_1, e_2, \ldots, e_n \rangle \equiv \mathbf{cons}\ e_1\ (\mathbf{cons}\ e_2\ (\ldots(\mathbf{cons}\ e_n\ \mathbf{nil})\ldots)).$$

Recursion

Non-termination

What happens when we evaluate?

$$\Omega = (\lambda h.(h\ h))(\lambda h.(h\ h))$$

Recursion

Self-reference is the essence of recursion.

U Combinator

$$\mathbf{U} = \lambda h.(h\ h)$$

$$\Omega = \mathbf{U}(\mathbf{U})$$

Factorial

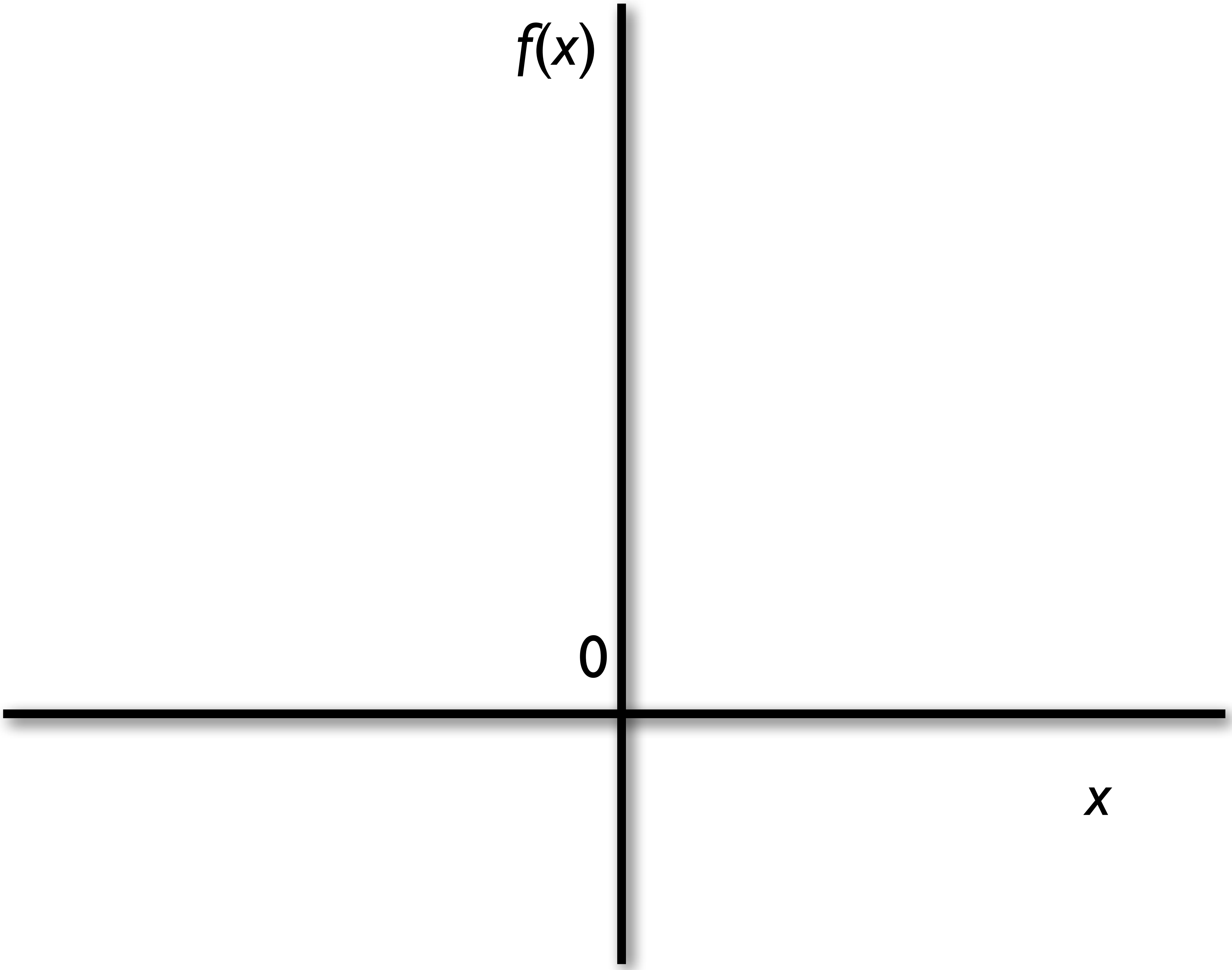
$$fact_{\mathbf{U}} = \mathbf{U}(\lambda h.\lambda n.\mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times (h\ h)(n-1))$$

A little more elegance

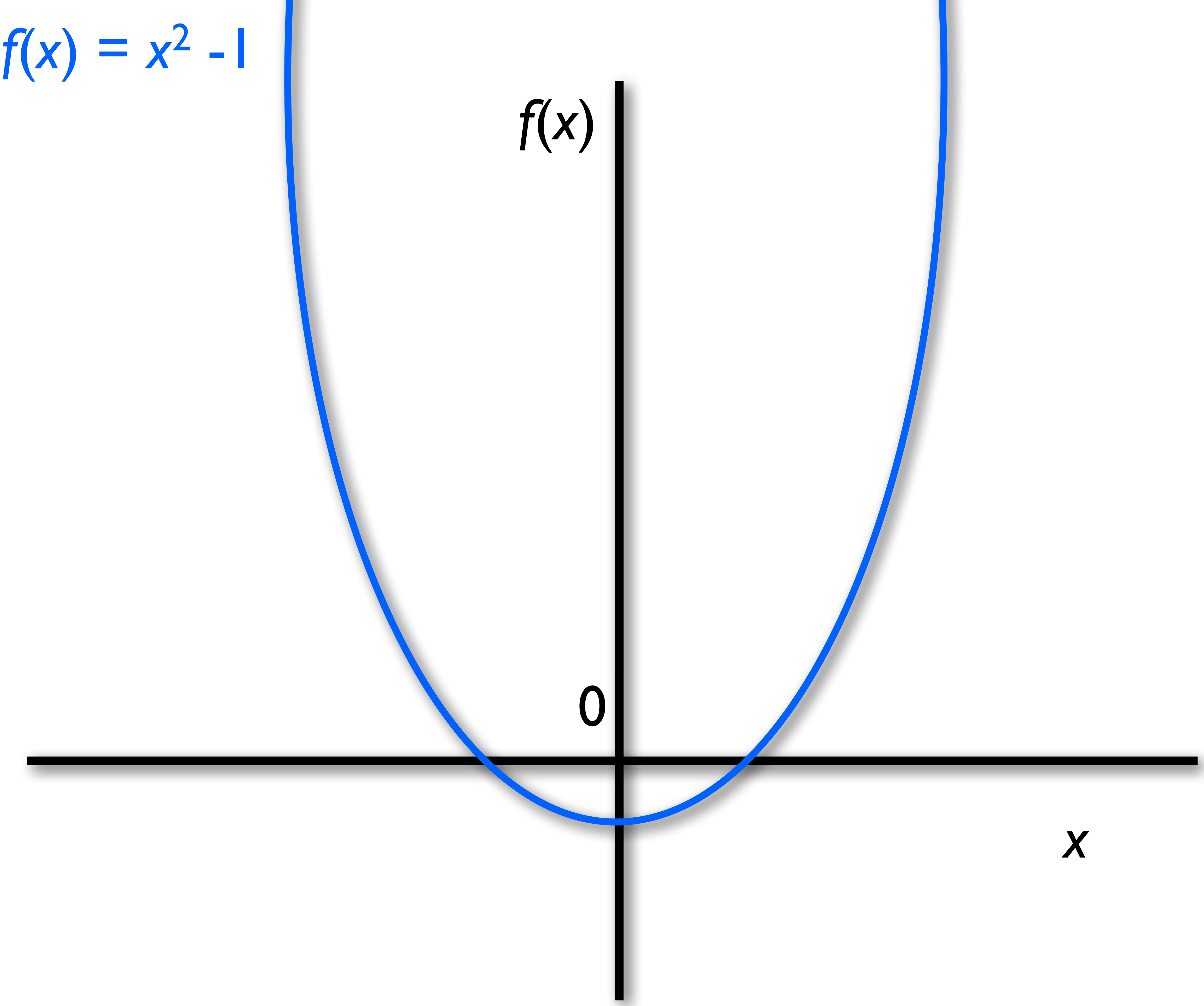
If $x = f(x)$, the point x is a **fixed point** of the function f.

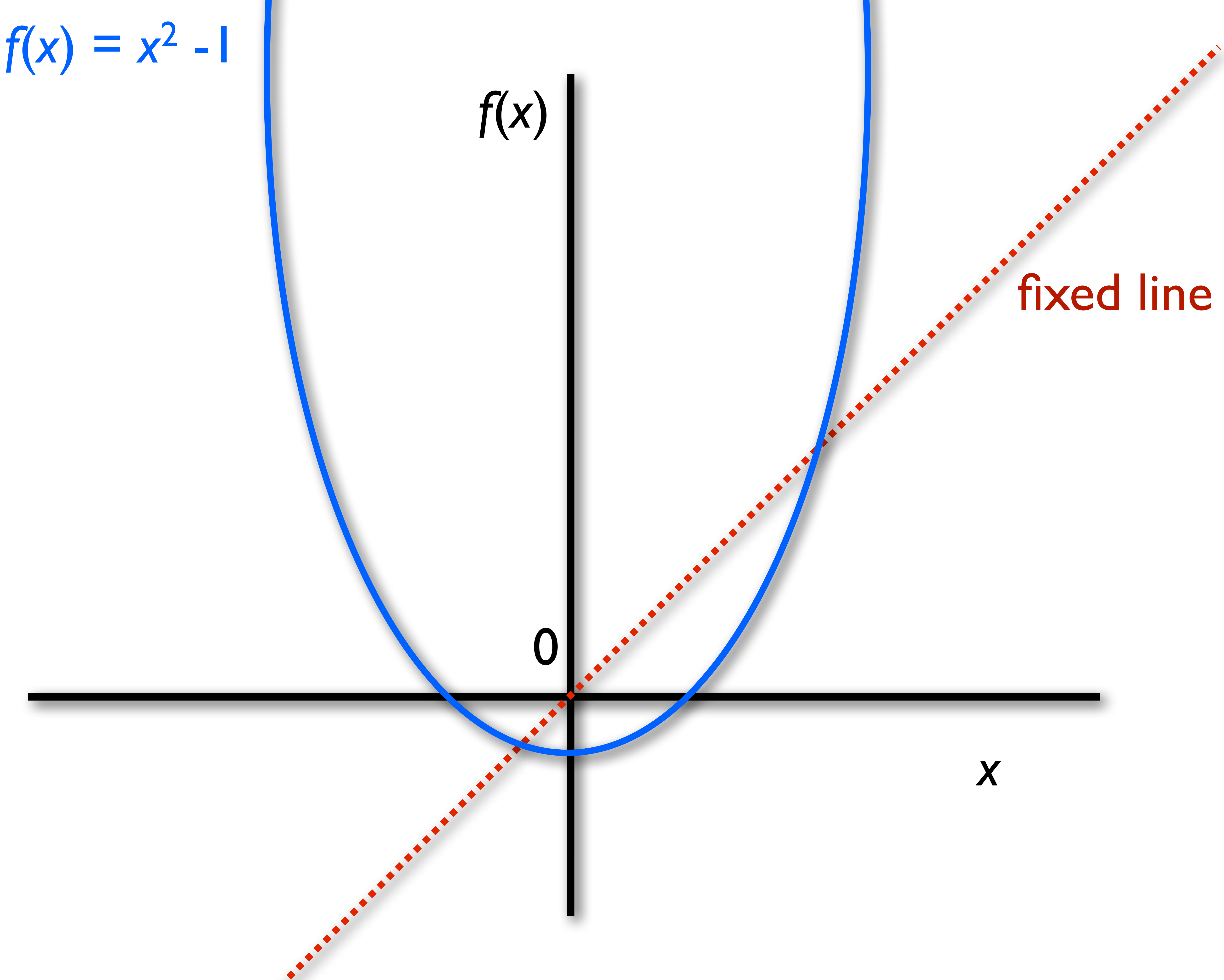
Algebra

- $x = x^2 - 1$ is a recursive definition of x
- If $f(v) = v^2 - 1$, then $x = f(x)$.
- Solutions are the fixed points of f.


f(x)
0
x

$f(x) = x^2 - 1$




$f(x) = x^2 - 1$
$f(x)$
fixed line
0
x

Factorial again

Factorial again

$$fact(n) = \mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times fact(n-1)$$

Factorial again

$$fact(n) = \textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

$$fact = \lambda n.\textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

Factorial again

$$fact(n) = \textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

$$fact = \lambda n.\textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

$$fact = F(fact)$$

Factorial again

$$fact(n) = \textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

$$fact = \lambda n.\textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times fact(n-1)$$

$$fact = F(fact)$$

$$F(f) = \lambda n.\textbf{if } (n \leq 0) \textbf{ then } 1 \textbf{ else } n \times f(n-1)$$

Fixed-point finder

- We want function Y that finds fixed points
- Technically, $Y(F) = x$, such that $F(x) = x$.
- Start off derivation with $Y(F) = F(Y(F))$.

Solving for **Y**

$$Y(F) = F(Y(F))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

$$Y = \mathbf{U}(\lambda h.\lambda F.F((h\,h)(F)))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

$$Y = \mathbf{U}(\lambda h.\lambda F.F((h\,h)(F)))$$

Does this work?

Solving for **Y**

$$Y(F) = F(Y(F))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

$$Y = \lambda F.F(\lambda x.(Y(F)(x)))$$

Solving for **Y**

$$Y(F) = F(Y(F))$$

$$Y = \lambda F.F(Y(F))$$

$$Y = \lambda F.F(\lambda x.(Y(F)(x)))$$

$$Y = \mathbf{U}(\lambda h.\lambda F.F(\lambda x.((h\ h)(F)(x))))$$

Y

$$\mathbf{Y} = (\lambda h.\lambda F.F(\lambda x.((h\ h)(F)(x))))(\lambda h.\lambda F.F(\lambda x.((h\ h)(F)(x))))$$

Factorial again

Factorial again

$$fact = F(fact)$$

$$F(f) = \lambda n.\mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times f(n-1)$$

Factorial again

$$fact = F(fact)$$

$$F(f) = \lambda n.\mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times f(n-1)$$

$$fact = \mathbf{Y}(F)$$

Factorial again

$$fact = F(fact)$$

$$F(f) = \lambda n.\mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times f(n-1)$$

$$fact = \mathbf{Y}(F)$$

$$fact = \mathbf{Y}(\lambda f.\lambda n.\mathbf{if}\ (n \leq 0)\ \mathbf{then}\ 1\ \mathbf{else}\ n \times f(n-1))$$

Experiment