Efficient control-flow analysis and beyond

Matt Might
University of Utah
matt.might.net

Clarifications

- *k*-CFA is not sole solution to constraints

For every polynomial-time algorithm you have,
there is an exponential algorithm I would rather run.

Alan Perlis

Objectives

- Two more popular CFAs
- Implementing small-step CFAs
- *k*-CFA for Java
- Abstract garbage collection
- Abstract counting
- Optimizations

Two more CFAs

poly/CFA (Wright *et al.*, 1998)

- Inspired by let-bound polymorphism
- Achieves effect of monomorphization
- Tends to be faster and more precise

poly/CFA (Wright *et al.*, 1998)

$$\widehat{Time} = \mathsf{Call} + \{0\}$$

$$\widehat{tick}(call, \hat{t}) = \begin{cases} call & \text{function is let-bound} \\ 0 & \text{otherwise.} \end{cases}$$

CPA (Agesen, 1995)

- CPA: Cartesian product algorithm
- Time = Product of argument information
- For example: Allocate types as time

CPA (Agesen, 1995)

$$\widehat{Time} = \mathsf{Type}^*$$

$$\widehat{tick}([\![(f\ e_1 : \tau_1 \cdots e_n : \tau_n)]\!], \hat{t}) = \langle \tau_1, \ldots, \tau_n \rangle$$

Implementing small-step 0CFA

Implementation

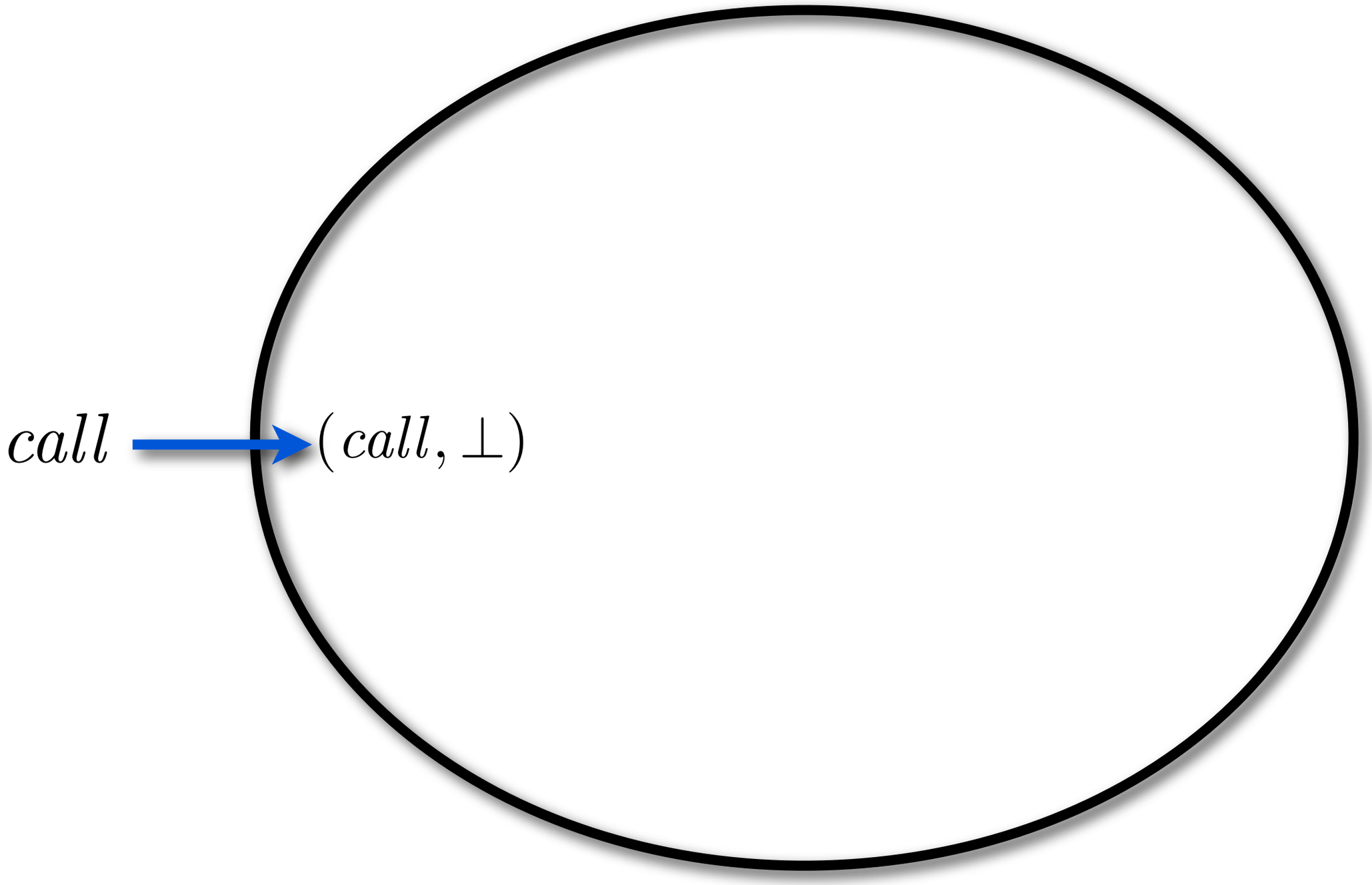
- Naïve implementation is straightforward
- Efficient implementation needs thought

Running 0CFA

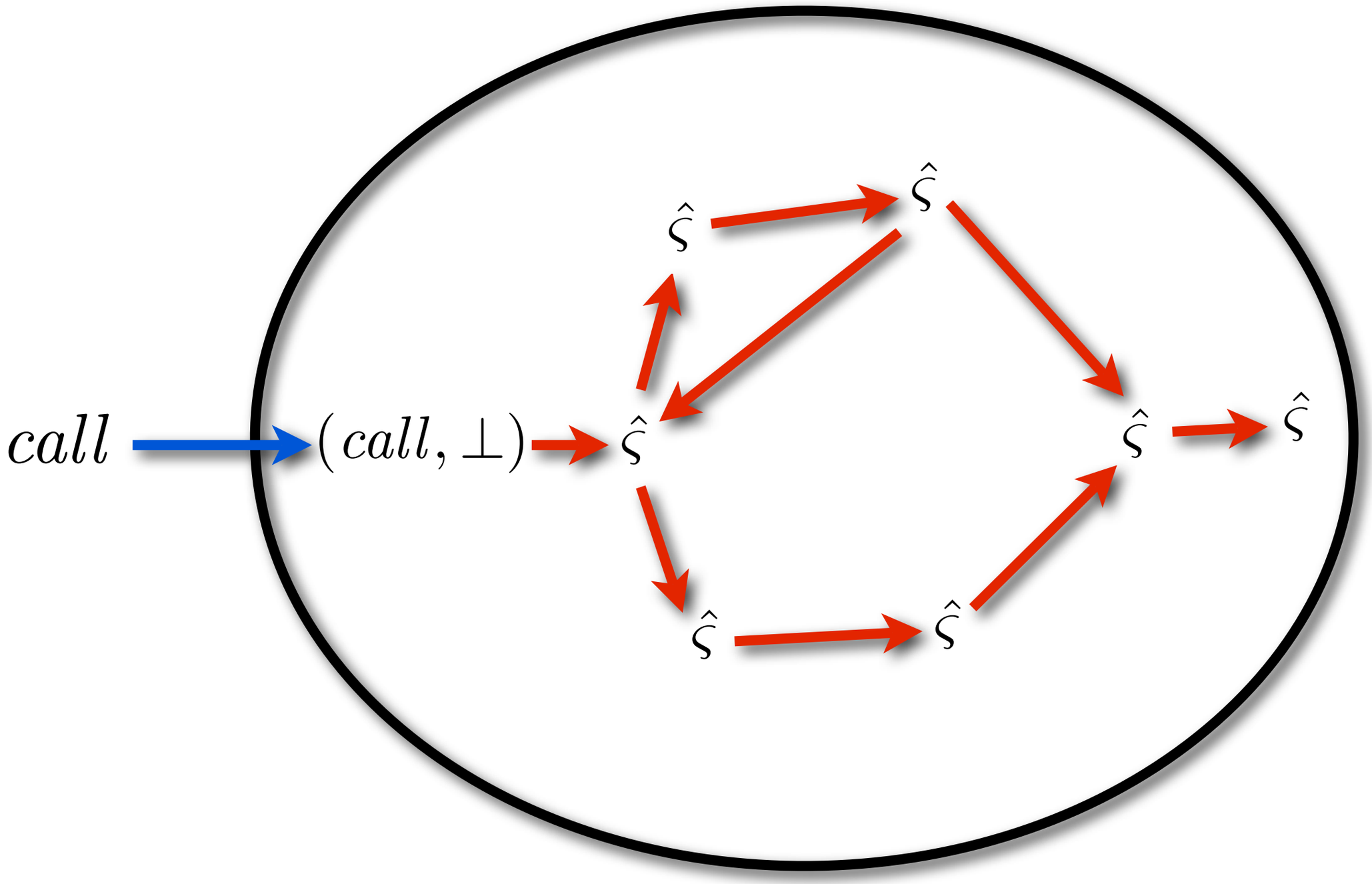
Running 0CFA

$$\hat{\Sigma}$$

Running 0CFA



Running 0CFA



Complexity of small-step 0CFA

Number of states: $|\mathsf{Call}| \cdot \left(2^{|\mathsf{Lam}|}\right)^{|\mathsf{Var}|}$

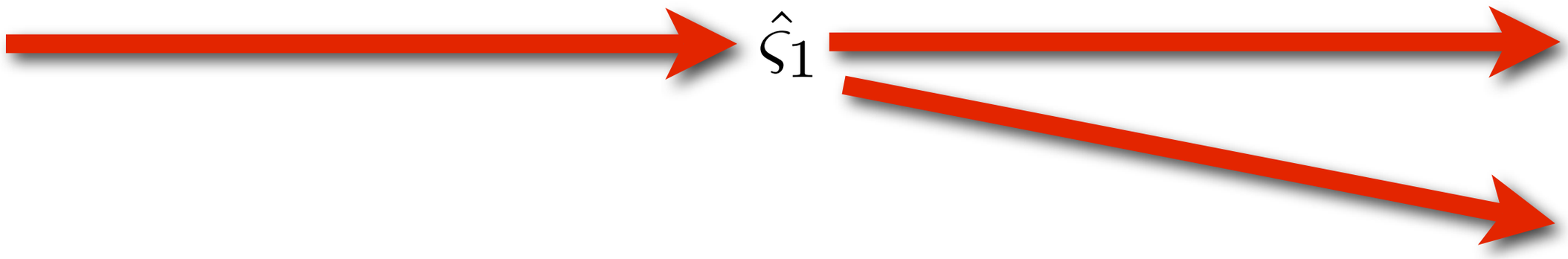
Techniques

- Exploiting monotonicity
- Contracting the state-space

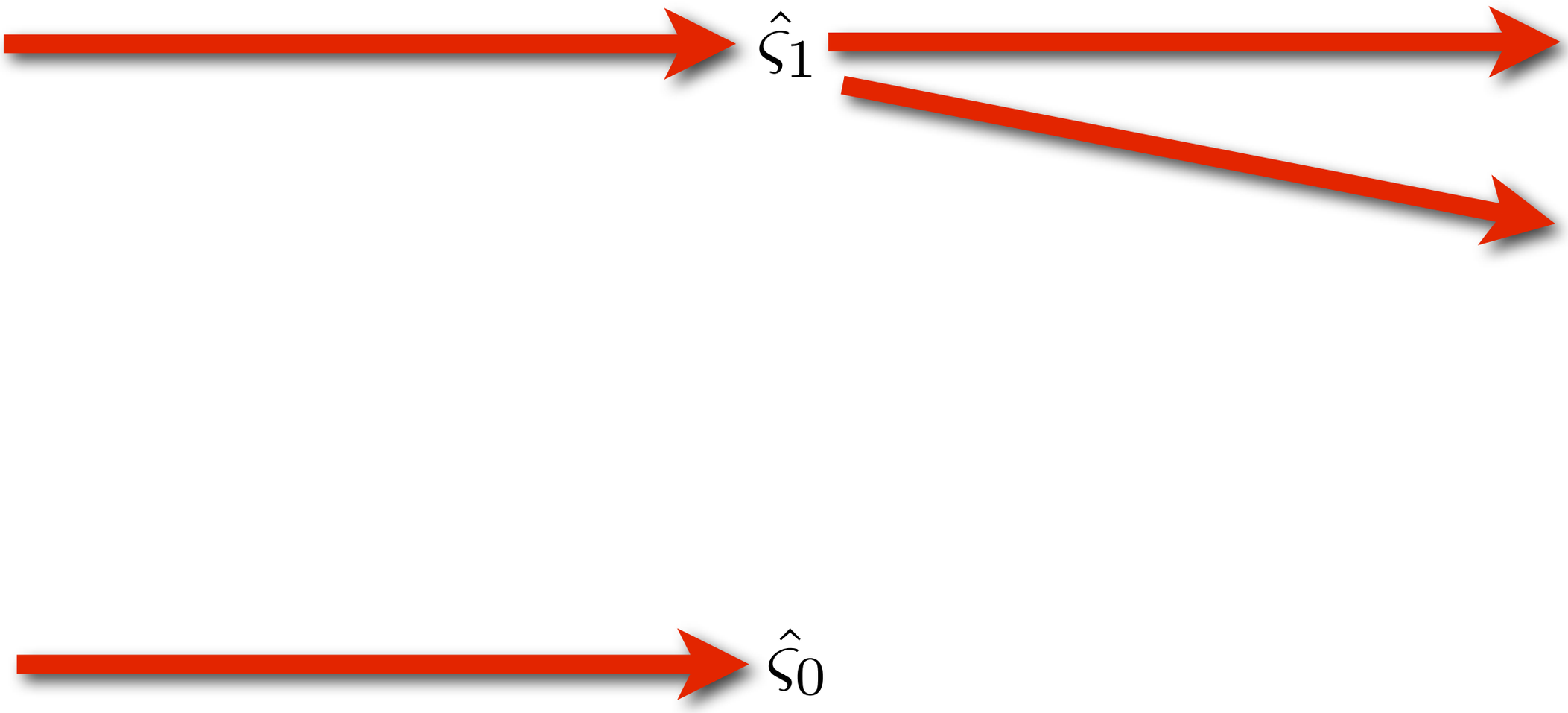
Exploiting monotonicity

$$
\begin{tikzcd}
\hat{s}_0 \arrow[r, "\rightsquigarrow"] \arrow[d, "\sqsubseteq"'] & \hat{s}_0' \arrow[d, "\sqsubseteq"] \\
\hat{s}_1 \arrow[r, dotted, "\rightsquigarrow"'] & \hat{s}_1'
\end{tikzcd}
$$

Exploiting monotonicity



Exploiting monotonicity



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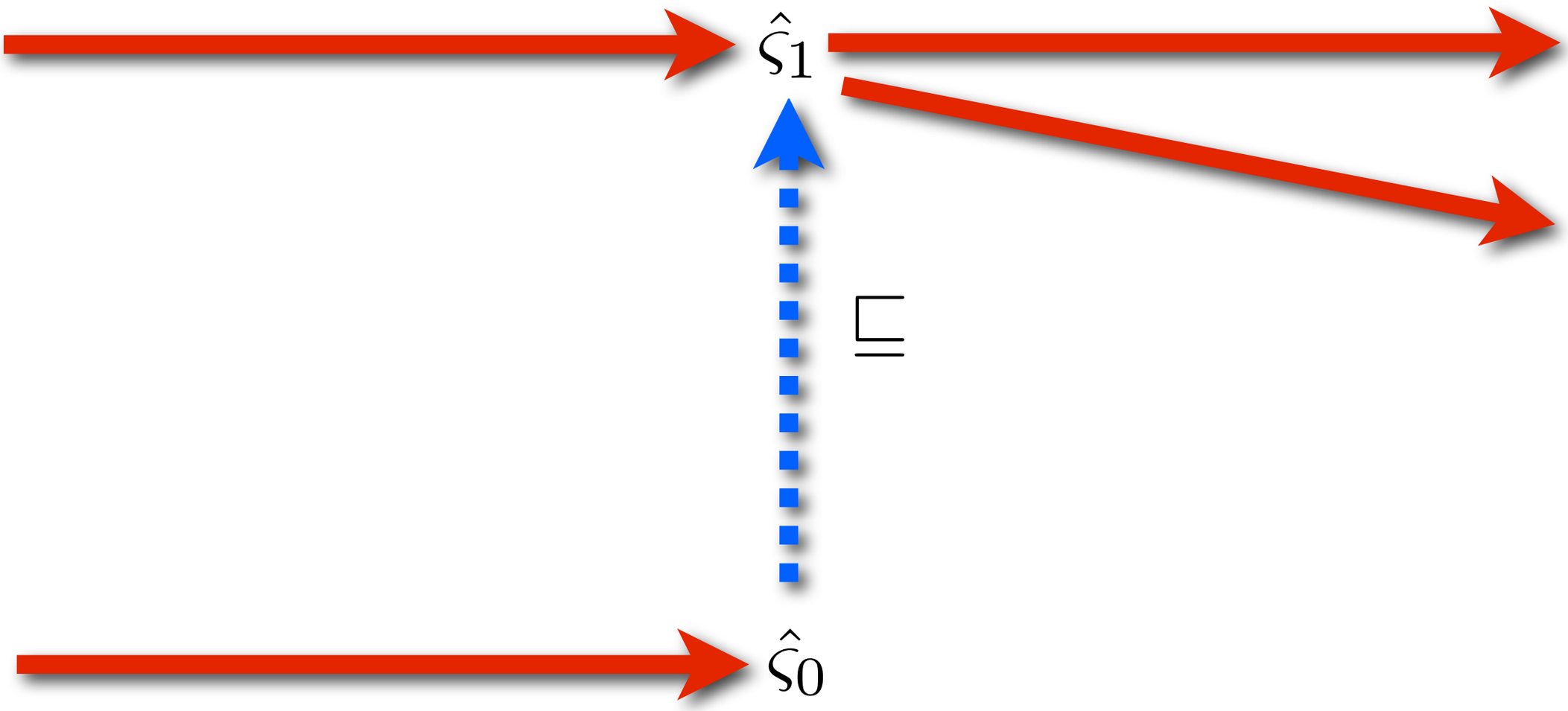

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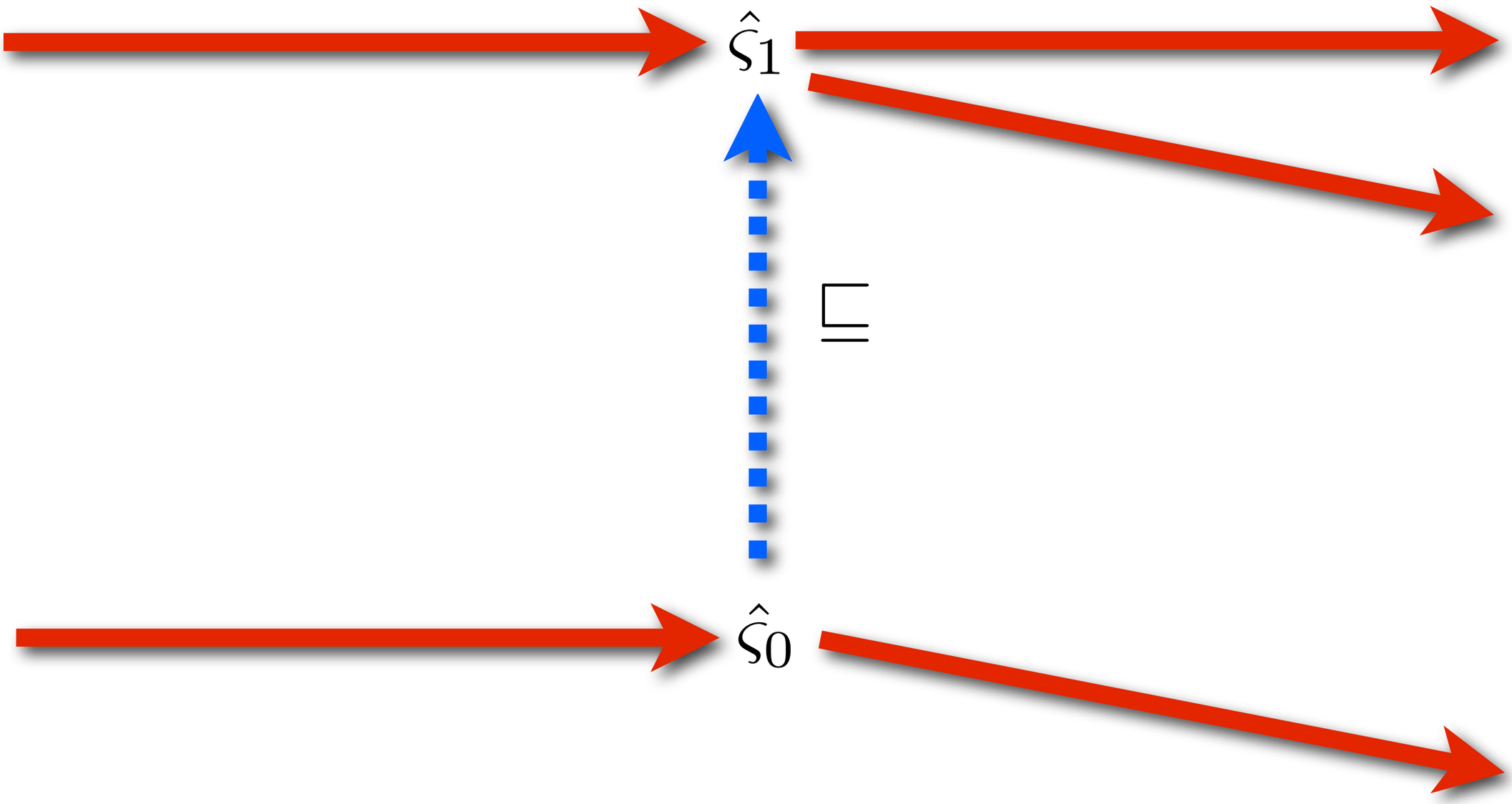
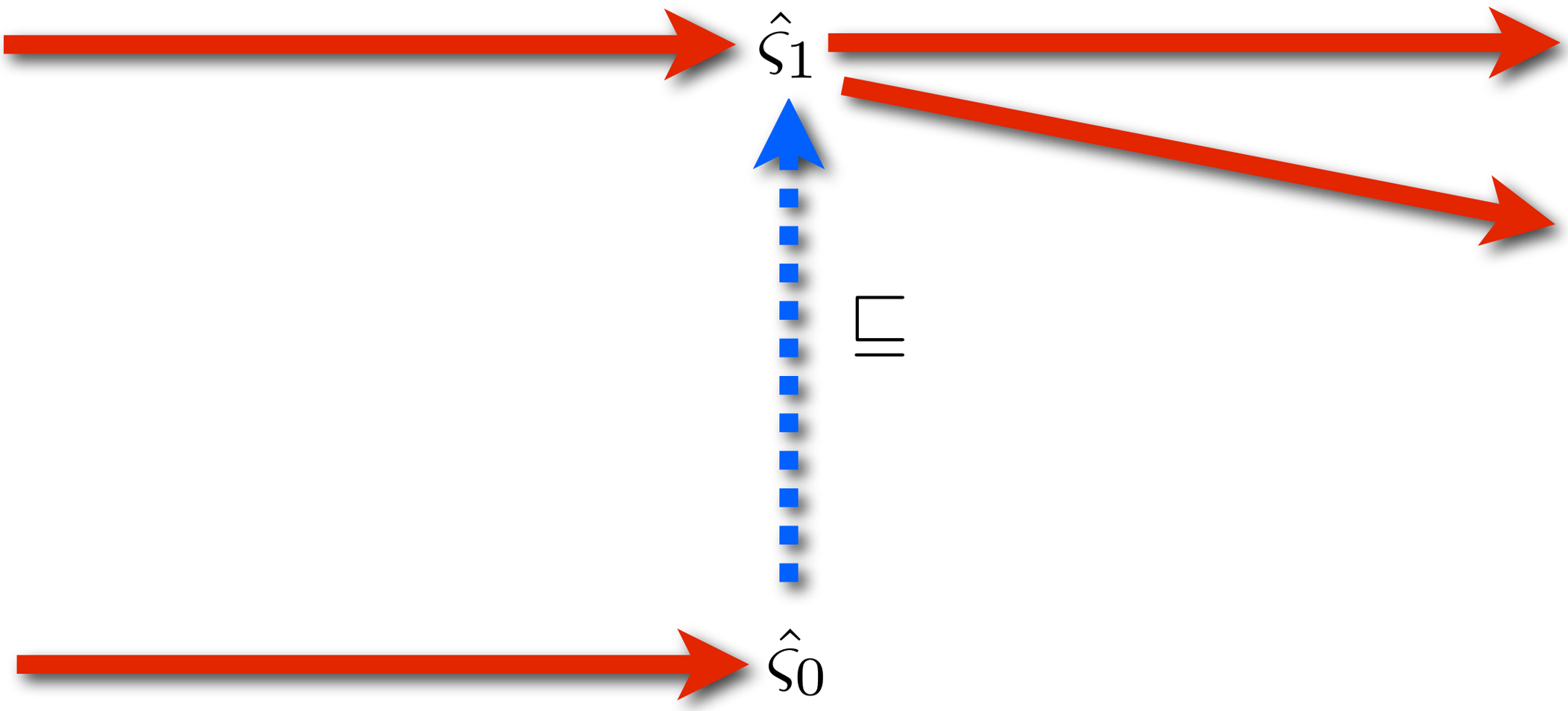


Exploiting monotonicity



Exploiting monotonicity

$$O\left(|\mathsf{Call}| \cdot \left(2^{|\mathsf{Lam}|}\right)^{|\mathsf{Var}|}\right)$$

System state-space

$$\Xi = \mathcal{P}(\Sigma)$$

System state-space

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$$f : \Xi \longrightarrow \Xi$$

System transition

$$f(\xi) = \{\varsigma' \mid \varsigma \in \xi \text{ and } \varsigma \Rightarrow \varsigma'\} \cup \{\varsigma_0\}$$

Abstract system

$$\hat{\Xi} = \mathcal{P}(\hat{\Sigma})$$

$$\hat{f} : \hat{\Xi} \longrightarrow \hat{\Xi}$$

Abstract transition

$$\hat{f}(\hat{\xi}) = \{\hat{\varsigma}' \mid \hat{\varsigma} \in \hat{\xi} \text{ and } \hat{\varsigma} \rightsquigarrow \hat{\varsigma}'\} \cup \{\hat{\varsigma}_0\}$$

0CFA system

$$\hat{\Xi} = \mathcal{P}(\mathsf{Call} \times \widehat{Store})$$

0CFA system

$$\hat{\Xi} = \mathcal{P}(\mathsf{Call}) \times \widehat{Store}$$

Monotonic transition

$$\hat{f}(C, \hat{\sigma}) = (C' \cup C, \hat{\sigma}'), where$$

$$S' = \left\{ \hat{\varsigma}' : call \in C \text{ and } (call, \hat{\sigma}) \rightsquigarrow \hat{\varsigma}' \right\}$$

$$C' = \left\{ call : (call, _) \in S' \right\}$$

$$\hat{\sigma}' = \bigsqcup_{(_, \hat{\sigma}) \in S'} \hat{\sigma}$$

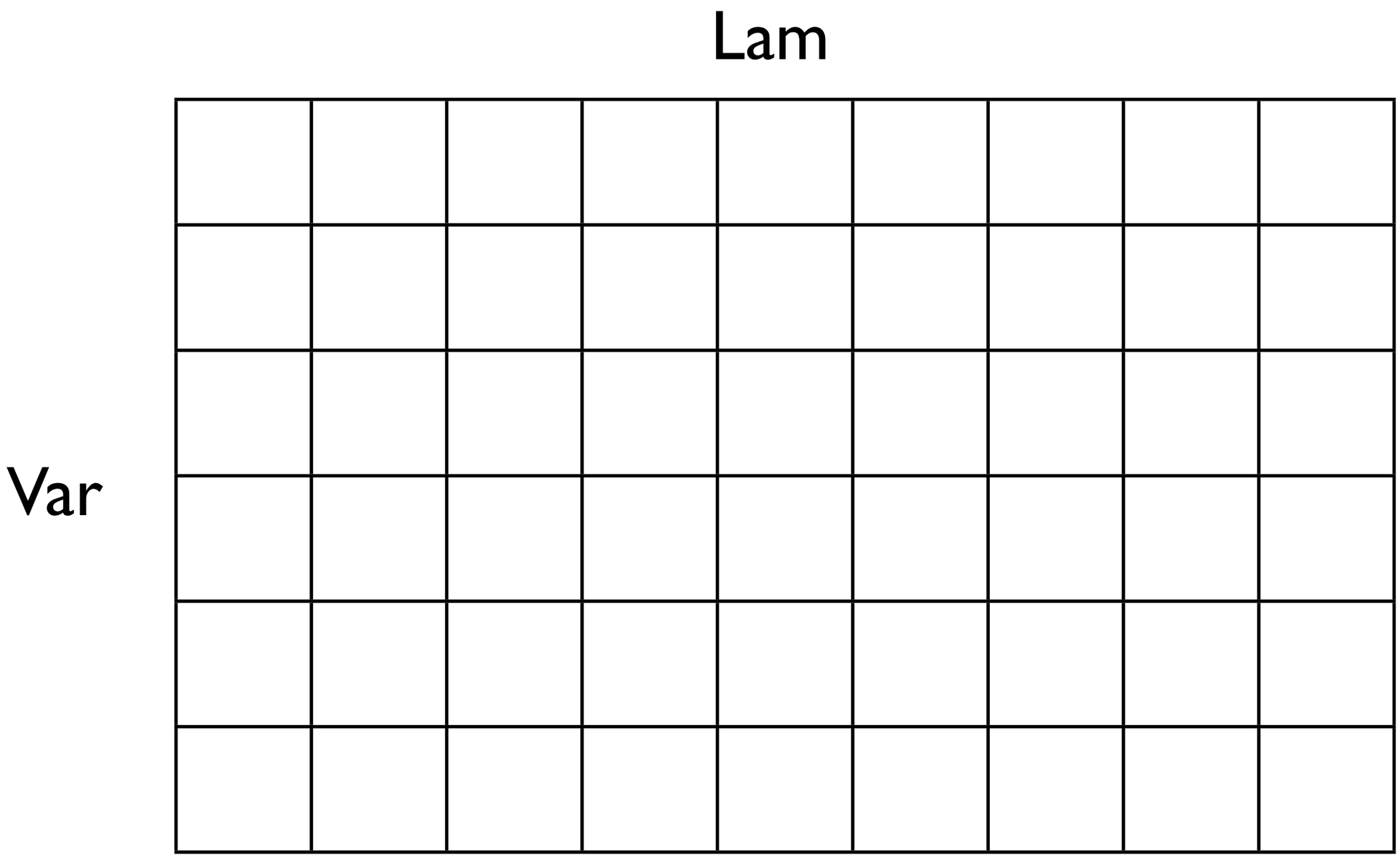
Flow-sensitive 0CFA

- Fixed point in $O(|\mathsf{Call}| \cdot |\mathsf{Var}| \cdot |\mathsf{Lam}|)$
- Analysis is flow-sensitive

Flow-insensitive 0CFA

- Start with state $\hat{\xi}_0 = (\text{Call}, \perp)$
- Total complexity is cubic

Cubic 0CFA visualized



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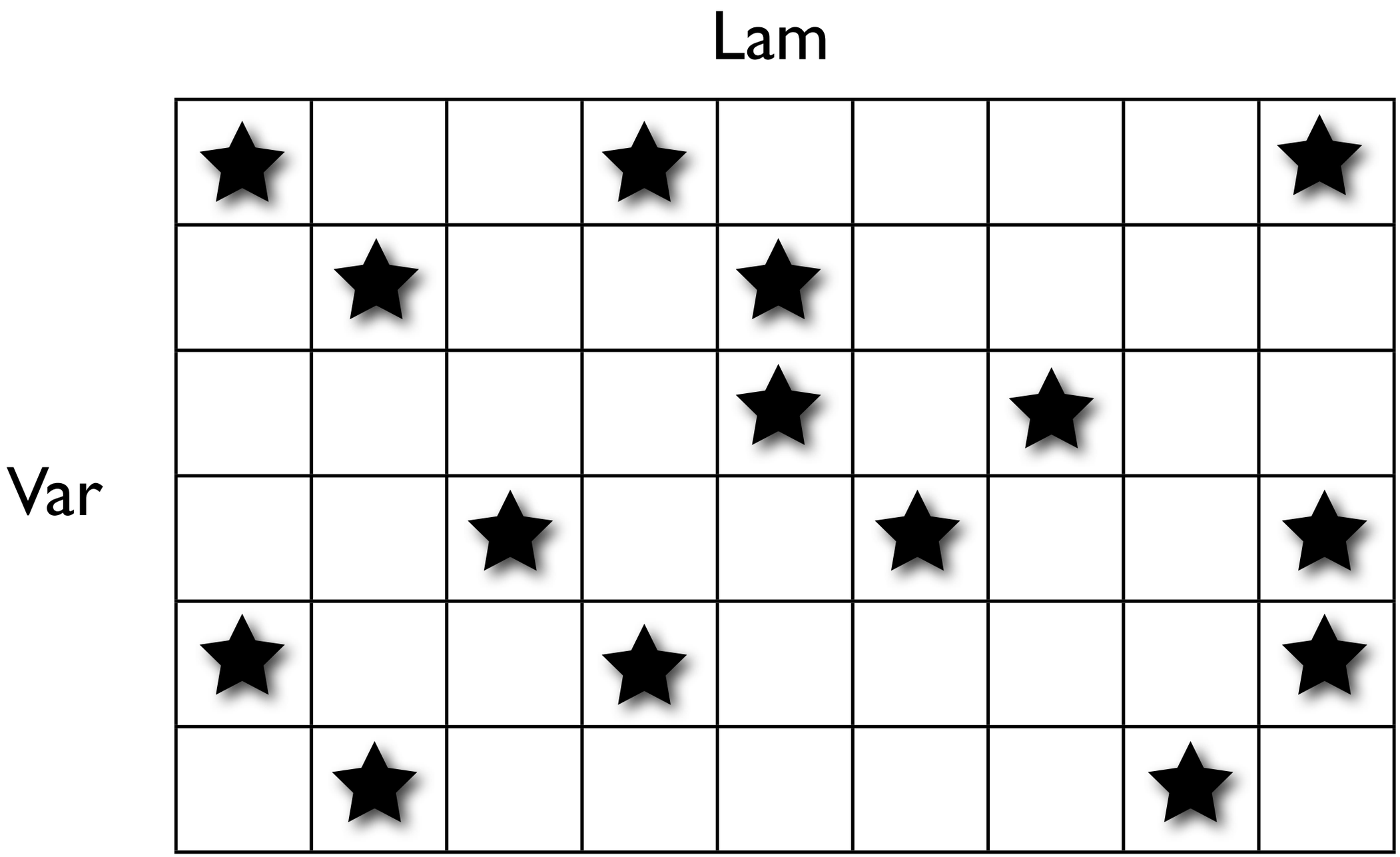

Per-point 0CFA

- Use abstract system $\hat{\Xi} = \text{Call} \rightarrow \widehat{Store}$
- Total complexity is quartic
- Often faster than cubic 0CFA
- Much more precise than cubic 0CFA

Theory and practice

"Expensive" 0CFA often faster than "efficient" 0CFA.

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Why?

Theory and practice

"Expensive" 0CFA often faster than "efficient" 0CFA.

Why?

Precise flow graphs are very sparse.

Implementing 1CFA

System-space for 1CFA

$$\hat{\Xi} = \mathcal{P}(\mathsf{Call} \times \widehat{BEnv} \times \widehat{Store})$$

System-space for 1CFA

$$\hat{\Xi} = \mathcal{P}(\mathsf{Call} \times \widehat{BEnv}) \times \widehat{Store}$$

System-space for 1CFA

$\hat{\Xi} = \mathsf{Call} \to \mathcal{P}(\widehat{} \times \widehat{})$

System-space for 1CFA

$\widehat{BEnv}$

Call

$\widehat{Clo}$

$\widehat{Addr}$

System-space for 1CFA

$$\mathsf{Call} \times \widehat{BEnv} + \widehat{Addr} \times \widehat{Clo}$$

$$= |\mathsf{Call}| \times (|\mathsf{Var}| \times |\mathsf{Call}|)^{|\mathsf{Var}|} + \left(|\mathsf{Lam}| \times (|\mathsf{Var}| \times |\mathsf{Call}|)^{|\mathsf{Var}|}\right)^{|\mathsf{Var}| \times |\mathsf{Call}|}$$

Excursion:

k-CFA for Java

Featherweight Java (Igarashi et al., 1999)

$$\texttt{L} ::= \texttt{class C extends C \{}\overline{\texttt{C}}\ \overline{\texttt{f}}\texttt{; K }\overline{\texttt{M}}\texttt{\}}$$

$$\texttt{K} ::= \texttt{C(}\overline{\texttt{C}}\ \overline{\texttt{f}}\texttt{)\{super(}\overline{\texttt{f}}\texttt{); this.}\overline{\texttt{f}}\texttt{=}\overline{\texttt{f}}\texttt{;\}}$$

$$\texttt{M} ::= \texttt{C m(}\overline{\texttt{C}}\ \overline{\texttt{x}}\texttt{)\{ return e; \}}$$

$$\texttt{e} ::= \texttt{x} \mid \texttt{e.f} \mid \texttt{e.m(}\overline{\texttt{e}}\texttt{)} \mid \texttt{new C(}\overline{\texttt{e}}\texttt{)} \mid \texttt{(C)e}$$

Abstract state-space

$$\hat{\varsigma} \in \hat{\Sigma} = \widehat{\mathsf{VExp}} \times \widehat{BEnv} \times \widehat{Store} \times \widehat{KontPtr} \times \widehat{Time}$$

$$\hat{\beta} \in \widehat{BEnv} = \mathsf{Var} \rightharpoonup \widehat{Addr}$$

$$\hat{\sigma} \in \widehat{Store} = \widehat{Addr} \to \hat{D}$$

$$\hat{d} \in \hat{D} = \mathcal{P}\left(\widehat{Val}\right)$$

$$\widehat{val} \in \widehat{Val} = \widehat{Obj} + \widehat{Kont}$$

$$\hat{o} \in \widehat{Obj} = \mathsf{Class} \times \widehat{BEnv}$$

$$\hat{\kappa} \in \widehat{Kont} = \widehat{\mathsf{Ctxt}} \times \widehat{BEnv} \times \widehat{KontPtr}$$

$\hat{a} \in \widehat{Addr}$ is a **finite** set of addresses

$\hat{t} \in \widehat{Time}$ is a **finite** set of time-stamps

$$\hat{p}^{\hat{\kappa}} \in \widehat{KontPtr} \subseteq \widehat{Addr}$$

Abstract state-space

$$\hat{\varsigma} \in \hat{\Sigma} = \mathsf{Body} \times \widehat{BEnv} \times \widehat{Store} \times \widehat{KontPtr} \times \widehat{Time}$$

$$\hat{\beta} \in \widehat{BEnv} = \mathsf{Var} \rightharpoonup \widehat{Addr}$$

$$\hat{\sigma} \in \widehat{Store} = \widehat{Addr} \rightarrow \hat{\square}$$

$$\hat{\square} \in \hat{\square} = \mathcal{P}\left(\widehat{Val}\right)$$

$$\widehat{val} \in \widehat{Val} = \widehat{Clo} + \widehat{Kont}$$

$$\widehat{clo} \in \widehat{Clo} = \mathsf{Lam} \times \widehat{BEnv}$$

$$\hat{\kappa} \in \widehat{Kont} = \mathsf{Var} \times \mathsf{Body} \times \widehat{Env} \times \widehat{KontPtr}$$

$\square \in \widehat{Addr}$ is a **finite** set of addresses

$\hat{\square} \in \widehat{Time}$ is a **finite** set of time-stamps

$$\hat{\square} \in \widehat{KontPtr} \subseteq \widehat{Addr}$$

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$\hat{a} \in \widehat{Addr}$ is a **finite** set of addresses

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$$\hat{p} \in \widehat{KontPtr} \subseteq \widehat{Addr}$$

Abstract contexts

$$
\begin{aligned}
\hat{E} \in \widehat{\mathsf{VExp}} ::= \; & \hat{d} \mid v \\
\mid \; & \hat{E}\,\texttt{.}\,f \\
\mid \; & \hat{E}\,\texttt{.}\,m\texttt{(}\hat{E}_1 \ldots \hat{E}_n\texttt{)} \\
\mid \; & \texttt{new}\; C\texttt{(}\hat{E}_1, \ldots, \hat{E}_n\texttt{)} \\
\mid \; & \texttt{(}C\texttt{)}\; \hat{E}
\end{aligned}
$$

Abstract contexts

$$
\begin{aligned}
\hat{E} \in \widehat{\mathsf{VExp}} ::= \; & \hat{d} \mid v \\
\mid \; & \hat{E}\,.\,f \\
\mid \; & \hat{E}\,.\,m(\hat{E}_1 \ldots \hat{E}_n) \\
\mid \; & \texttt{new}\; C(\hat{E}_1, \ldots, \hat{E}_n) \\
\mid \; & (C)\; \hat{E}
\end{aligned}
$$

$$
\begin{aligned}
\hat{C} \in \widehat{\mathsf{Ctxt}} ::= \; & \square \\
\mid \; & \hat{C}\,.\,f \\
\mid \; & e\,.\,m(\hat{d}_1, \ldots, \hat{d}_n, \hat{C}, e_1, \ldots, e_m) \\
\mid \; & \hat{C}\,.\,m(\hat{d}_1, \ldots, \hat{d}_n) \\
\mid \; & \texttt{new}\; C(\hat{d}_1, \ldots, \hat{d}_n, \hat{C}, e_1, \ldots, e_m) \\
\mid \; & (C)\; \hat{C}
\end{aligned}
$$

Abstract semantics

$(\hat{C}[e], \hat{\beta}, \hat{\sigma}, \overset{\hat{\kappa}}{p}, \hat{t}) \rightsquigarrow (e, \hat{\beta}, \hat{\sigma}', \overset{\hat{\kappa}}{p}{}', \hat{t}')$, where

$$\hat{\kappa} = (\hat{C}, \hat{\beta}, \overset{\hat{\kappa}}{p})$$

$$\overset{\hat{\kappa}}{p}{}' = \widehat{alloc}_{\hat{\kappa}}(\ldots)$$

$$\hat{\sigma}' = \hat{\sigma} \sqcup [\overset{\hat{\kappa}}{p}{}' \mapsto \{\hat{\kappa}\}]$$

$$\hat{t}' = \widehat{tick}(\ldots)$$

Abstract semantics

$$
\begin{aligned}
(\llbracket \hat{d}_f\,.\,m(\hat{d}_1, \ldots, \hat{d}_n) \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\overset{\kappa}{p}}, \hat{t}) \rightsquigarrow (e, \hat{\beta}', \hat{\sigma}', \hat{\overset{\kappa}{p}}, \hat{t}'), &\text{ where} \\
\hat{\mathcal{M}} &: \hat{D} \times \mathsf{MethodCall} \to \mathcal{P}\,(\mathsf{Method}) \\
M &\in \hat{\mathcal{M}}(\hat{d}_f, m) \\
M &= \llbracket C'\ m\ (C_1\ v_1, \ldots, C_n\ v_n)\ e \rrbracket \\
\hat{a}_i &= \widehat{alloc}(v_i, \ldots) \\
\hat{\beta}' &= [\texttt{this} \mapsto \hat{d}_f, v_i \mapsto \hat{a}_i] \\
\hat{\sigma}' &= \hat{\sigma} \sqcup [\hat{a}_i \mapsto \hat{d}_i] \\
\hat{t}' &= \widehat{tick}(\ldots)
\end{aligned}
$$

Abstract semantics

$$(\llbracket \texttt{new}\ C\texttt{(}\hat{d}_1, \ldots, \hat{d}_n\texttt{)} \rrbracket, \hat{\beta}, \hat{\sigma}, \overset{\hat{\kappa}}{p}, \hat{t}) \rightsquigarrow (\hat{d}, \hat{\beta}, \hat{\sigma}', \overset{\hat{\kappa}}{p}, \hat{t}'), \text{ where}$$

$$(\hat{\mathcal{K}}, n, m) = \hat{\mathcal{C}}(C)$$

$$\hat{\mathcal{K}} : \hat{D}^n \times \widehat{Addr}^m \to (\widehat{Addr} \to \hat{D}) \times (\mathsf{Field} \to \widehat{Addr})$$

$$\hat{a}_i = \widehat{alloc}_i(\ldots)$$

$$(\Delta\hat{\sigma}, \hat{\beta}') = \hat{\mathcal{K}}(\hat{d}_1, \ldots, \hat{d}_n, \hat{a}_1, \ldots, \hat{a}_m)$$

$$\hat{\sigma}' = \hat{\sigma} \sqcup \Delta\hat{\sigma}$$

$$\hat{d} = \left\{ (C, \hat{\beta}') \right\}$$

$$\hat{t}' = \widehat{tick}(\ldots)$$

Abstract semantics

$$(\hat{d}, \hat{\beta}, \hat{\sigma}, \overset{\hat{\kappa}}{p}, \hat{t}) \rightsquigarrow (\hat{C}[\hat{d}], \hat{\beta}', \hat{\sigma}, \overset{\hat{\kappa}}{p}{}', \hat{t}'), \text{ where}$$

$$(\hat{C}, \hat{\beta}', \overset{\hat{\kappa}}{p}{}') \in \hat{\sigma}(\overset{\hat{\kappa}}{p})$$

$$\hat{t}' = \widehat{tick}(\ldots)$$

Abstract garbage collection

Main idea

- Garbage collect a concrete semantics
- Perform an abstract interpretation
- Watch precision and speed go up

Abstract GC Illustrated

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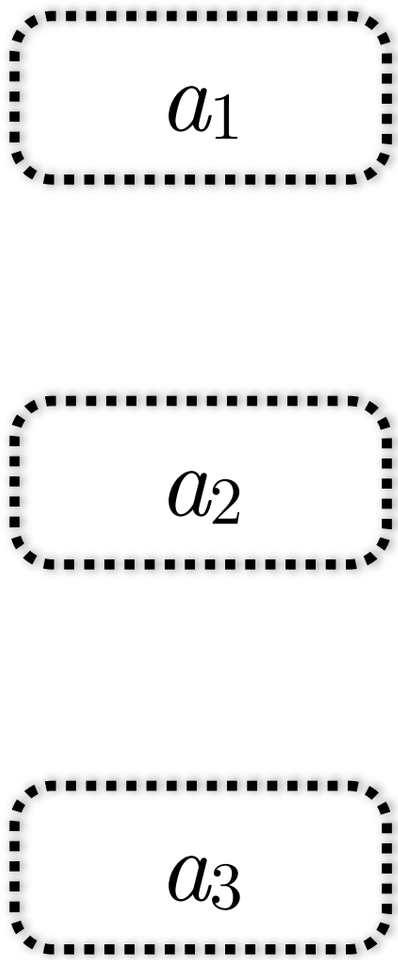

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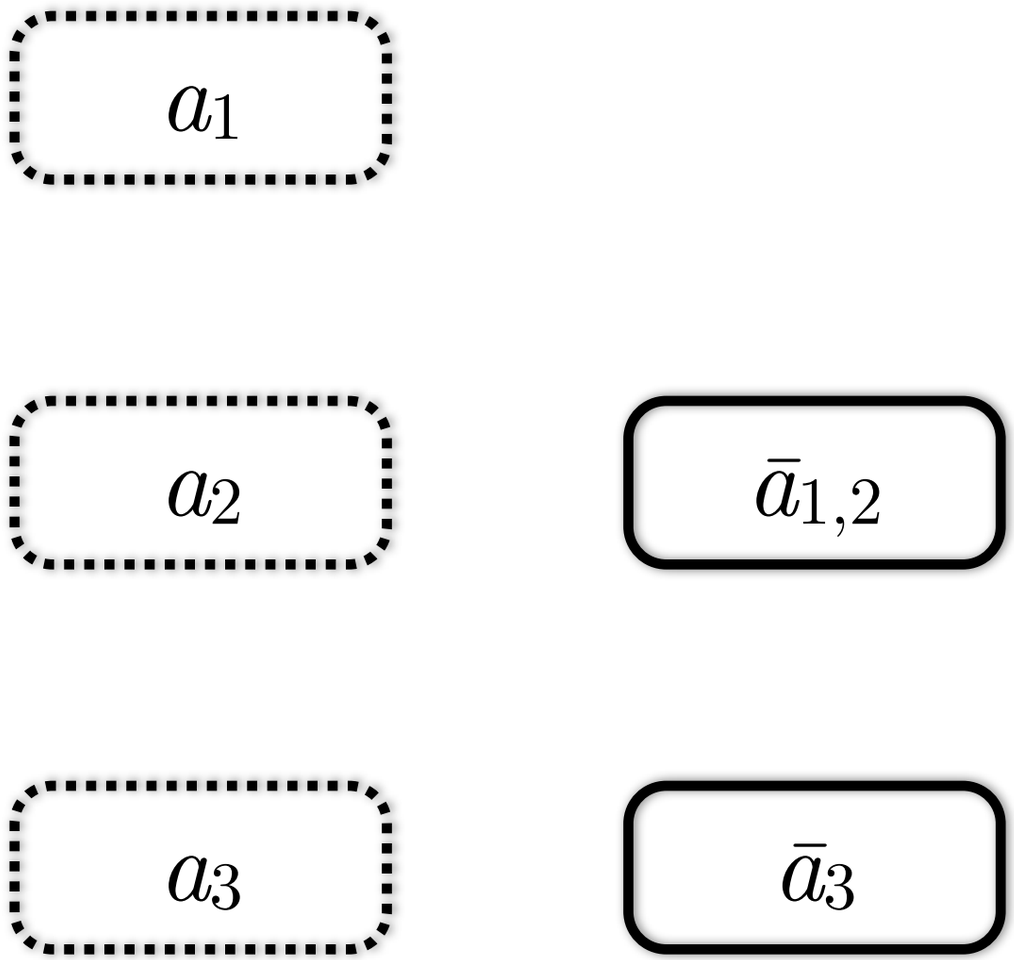

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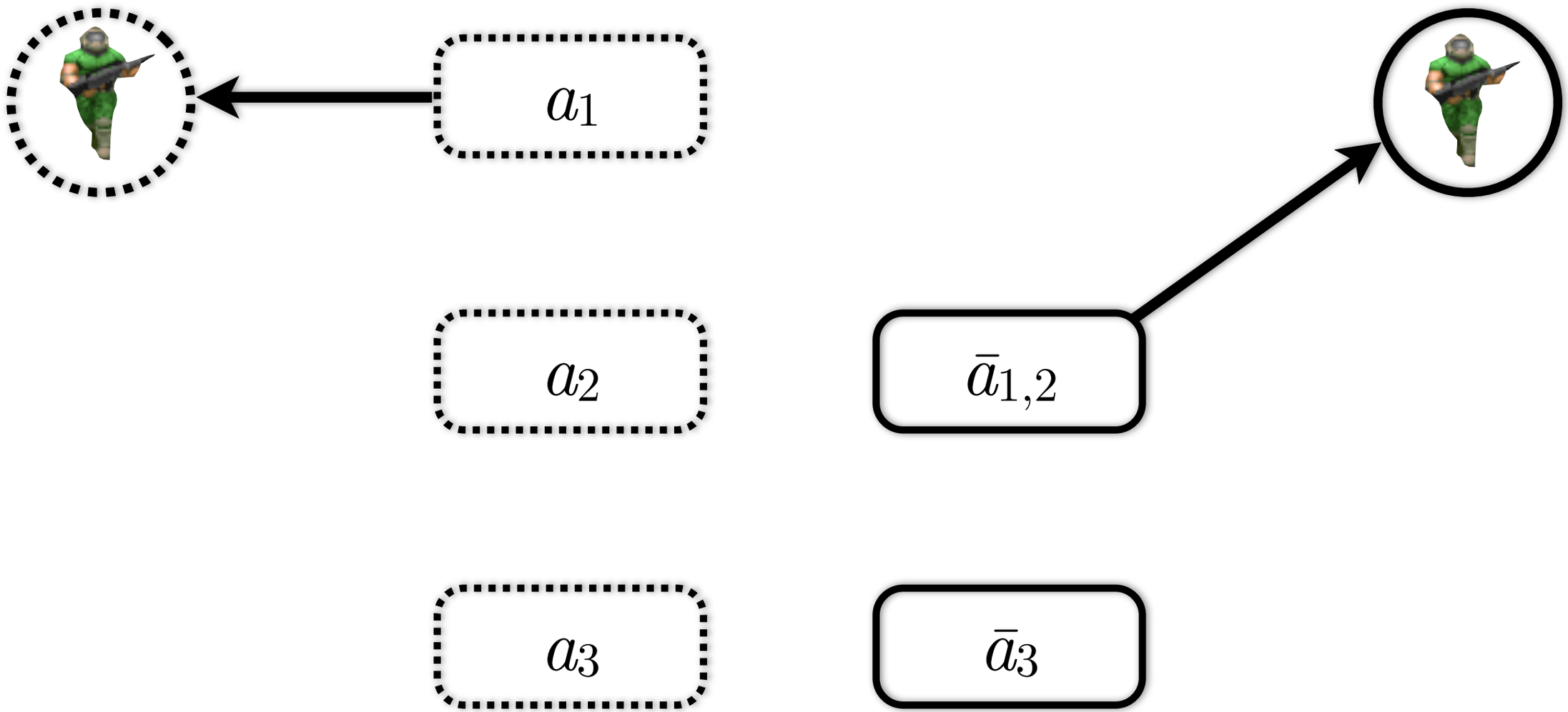

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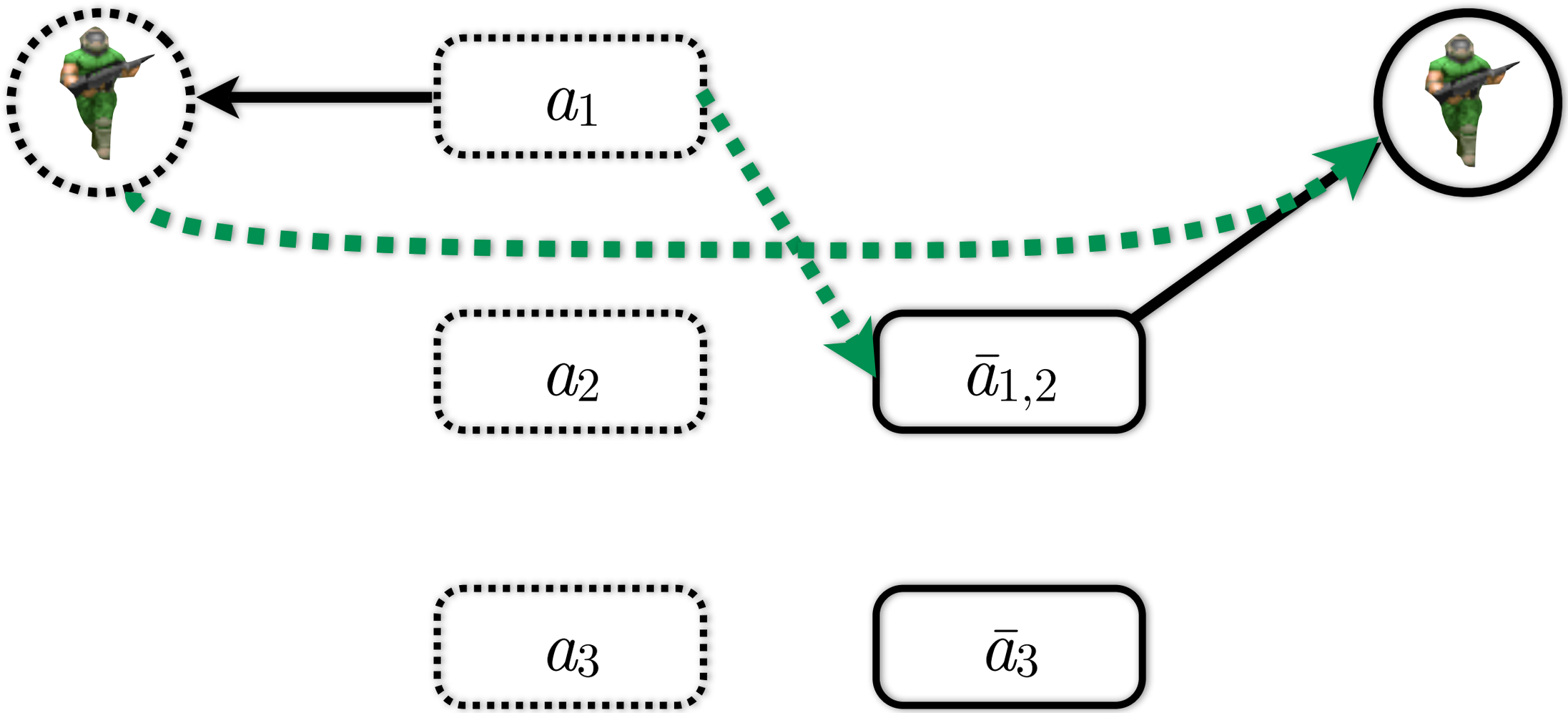

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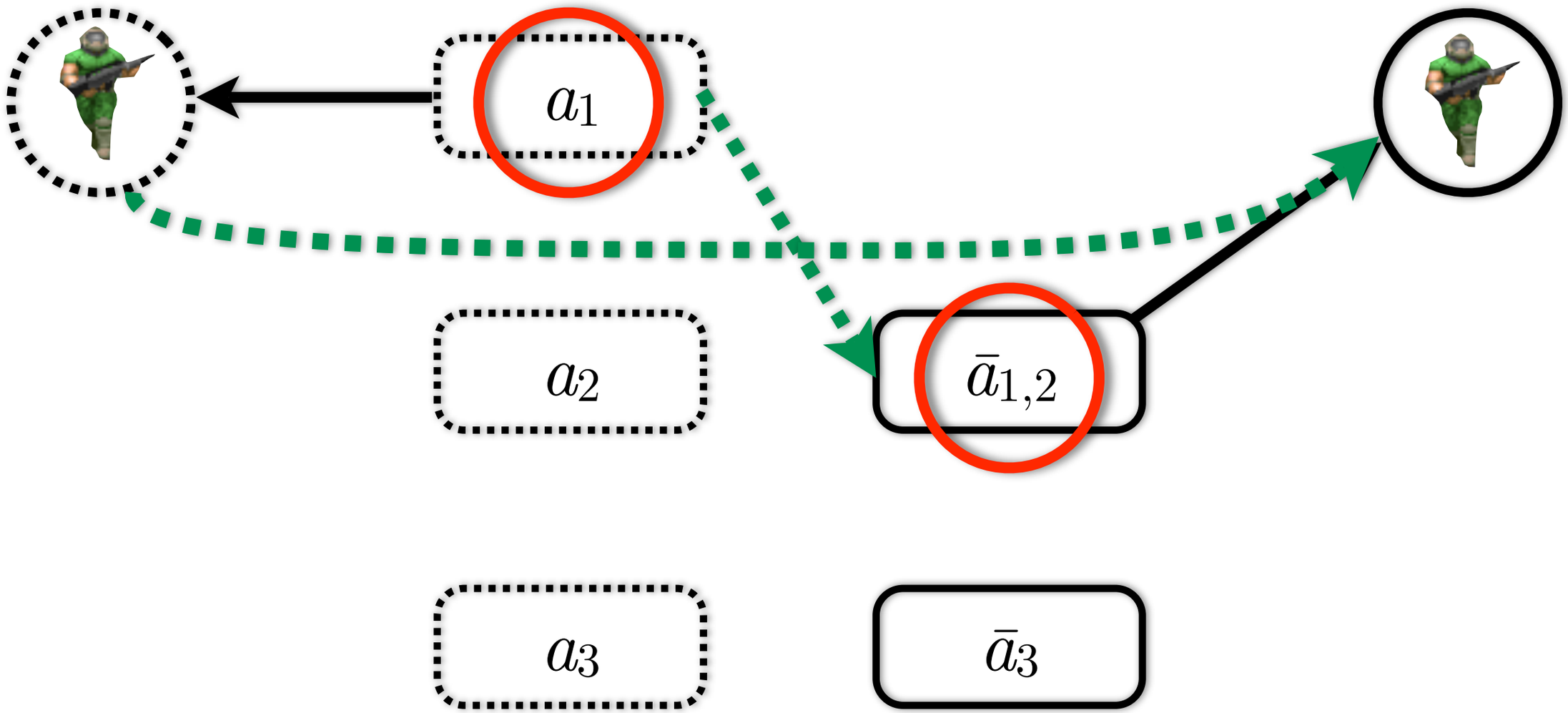

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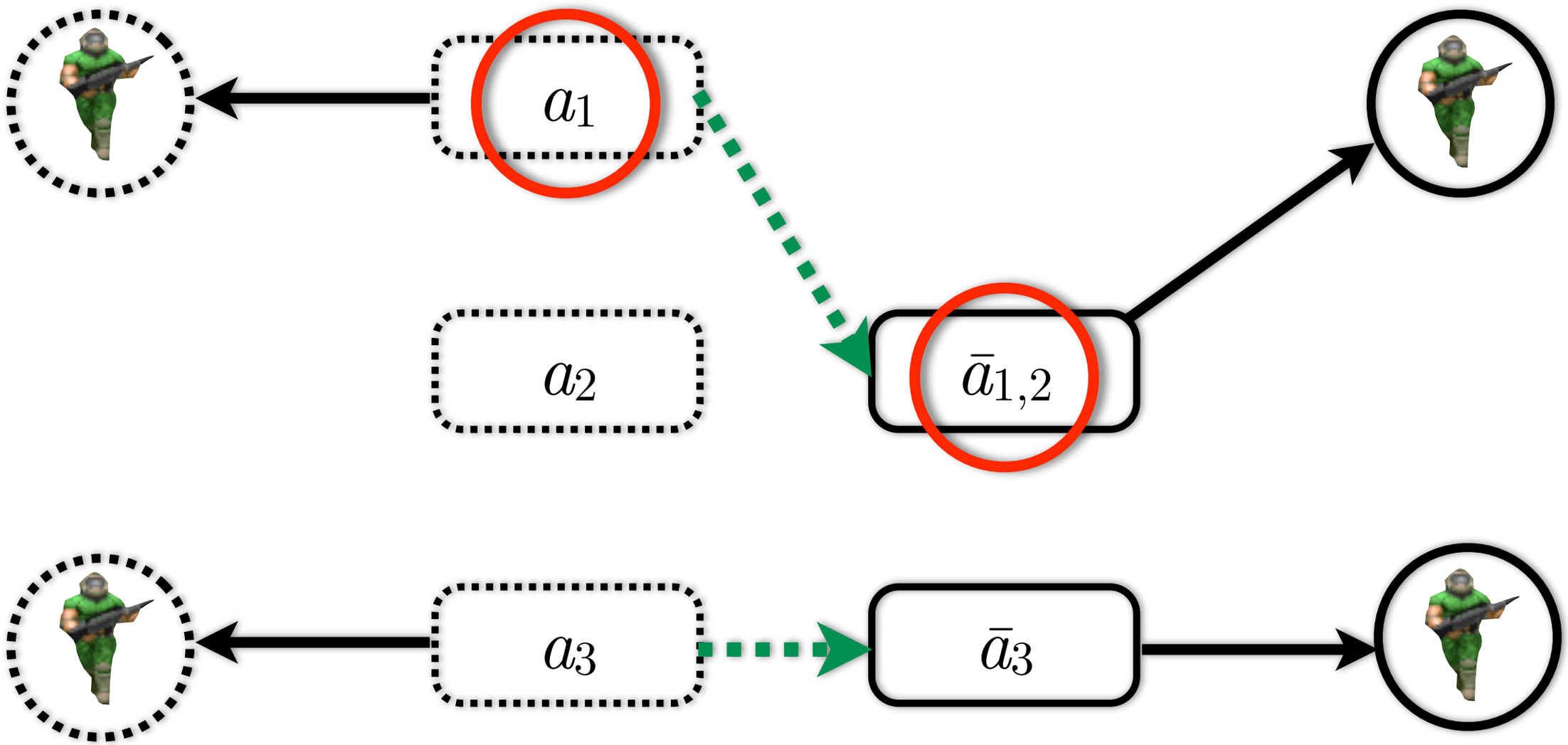

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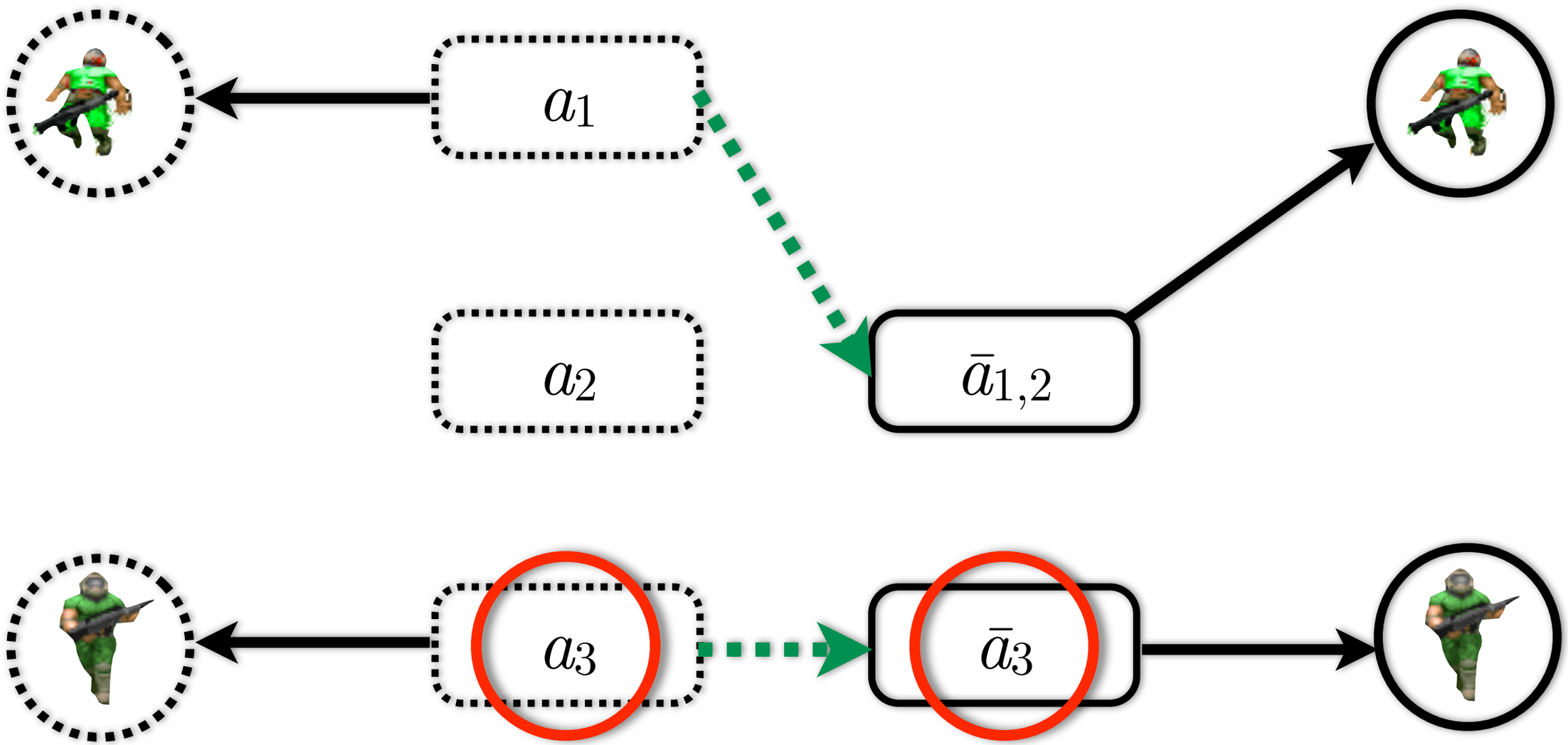

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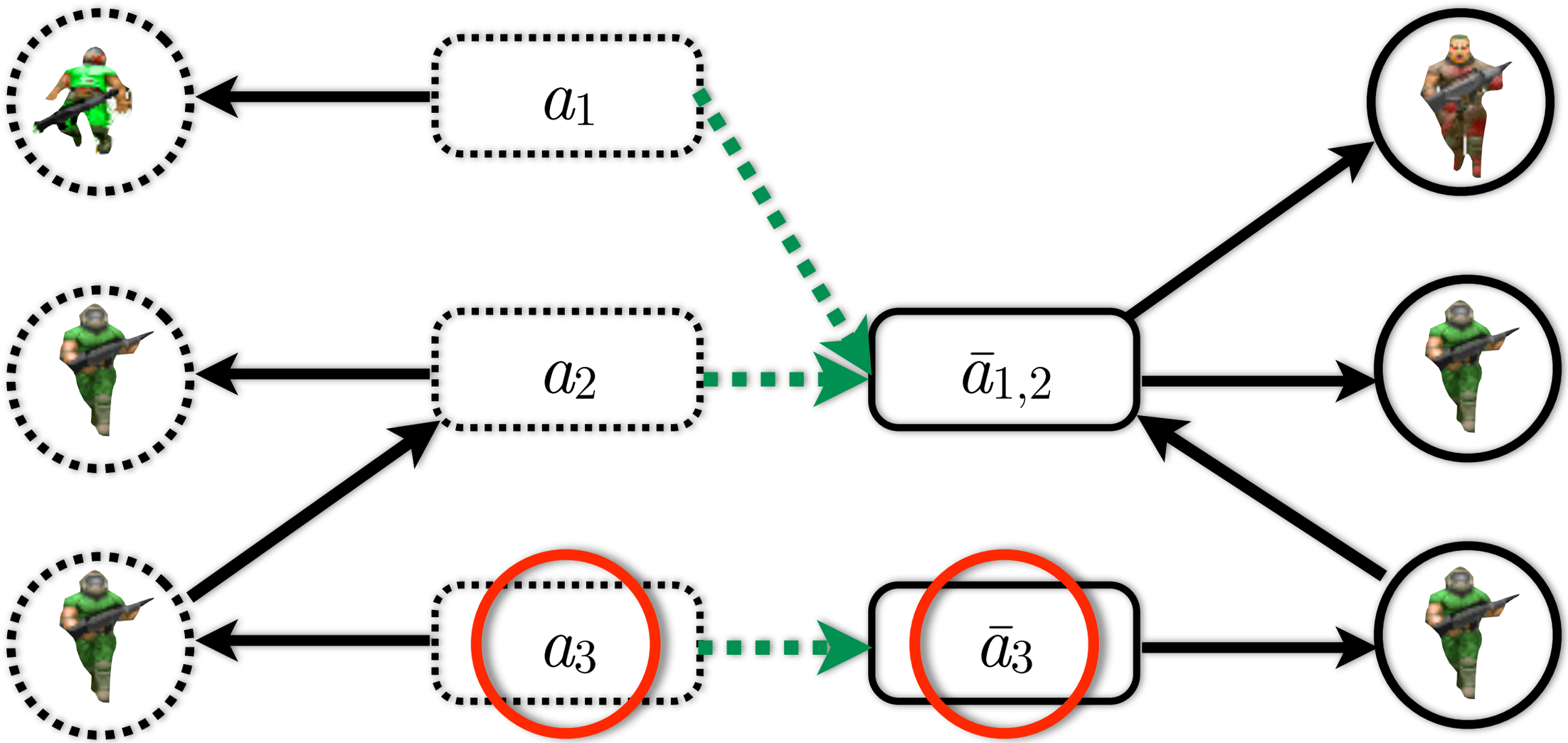

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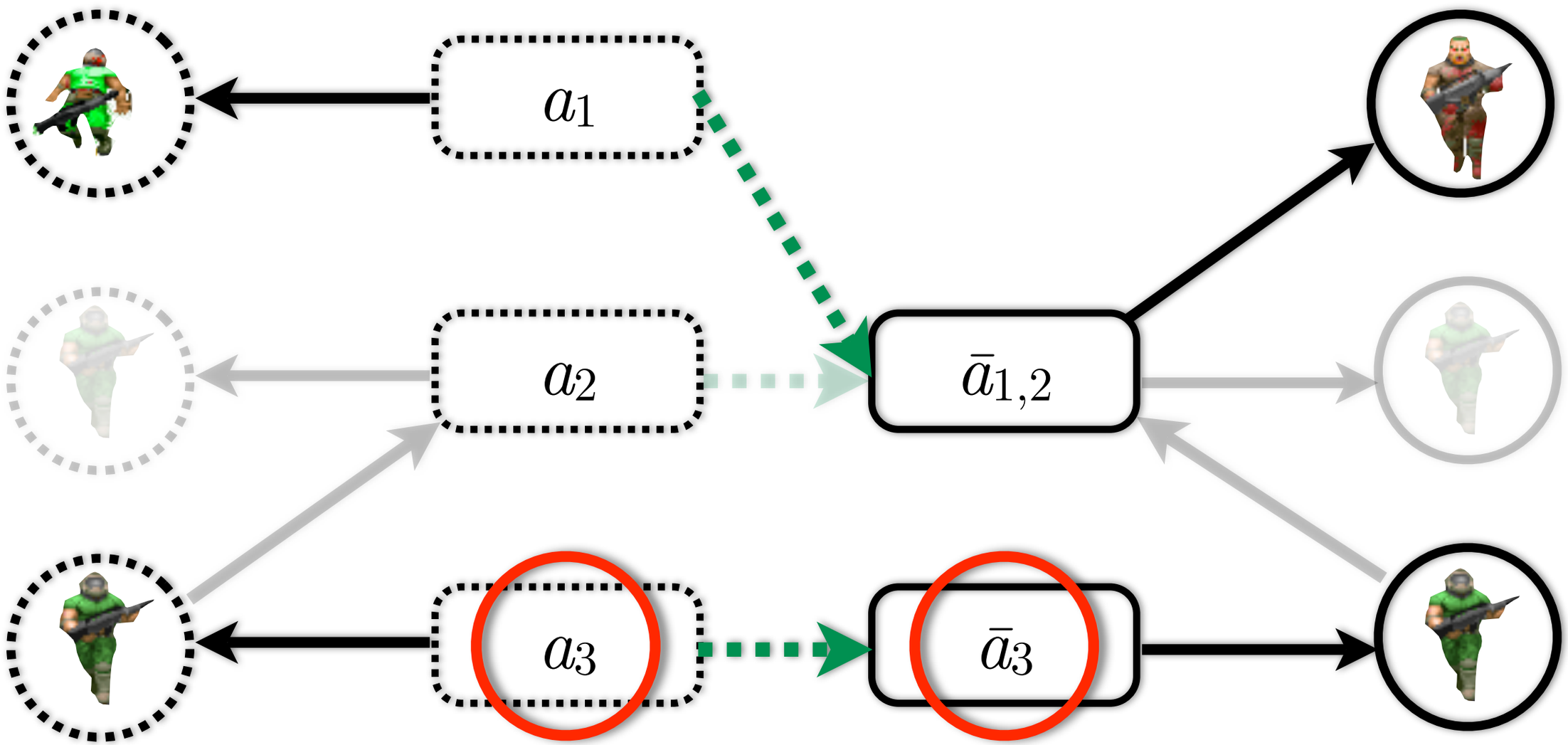

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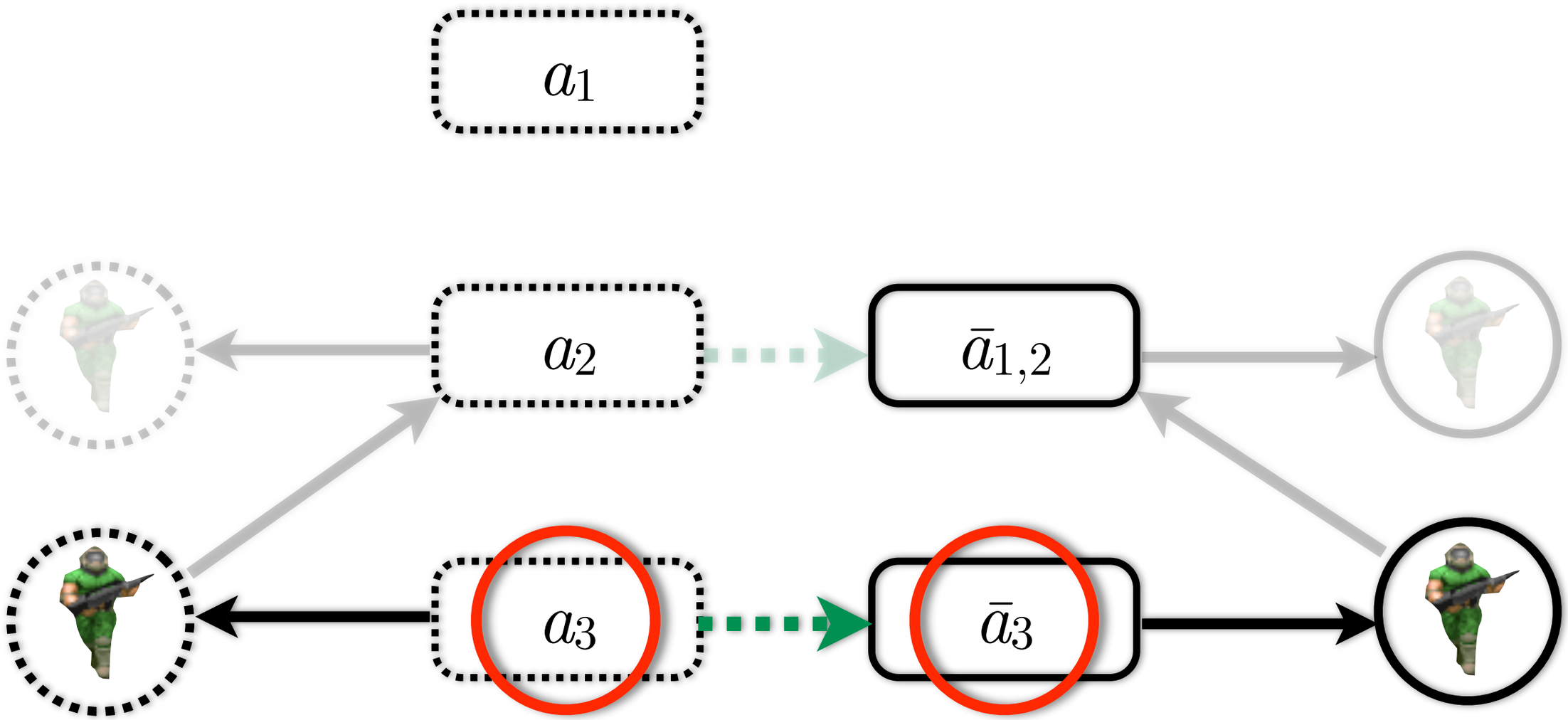

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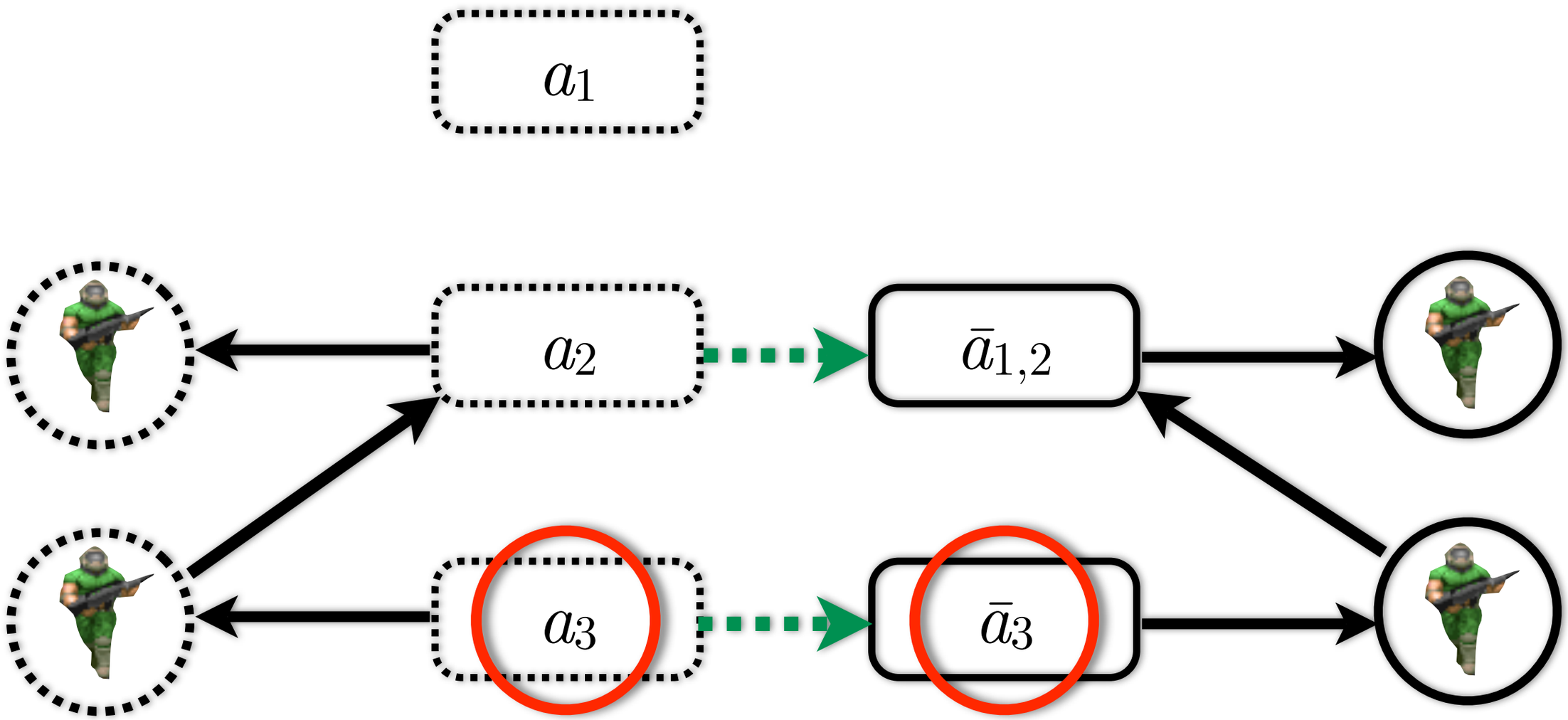

Effects of abstract GC

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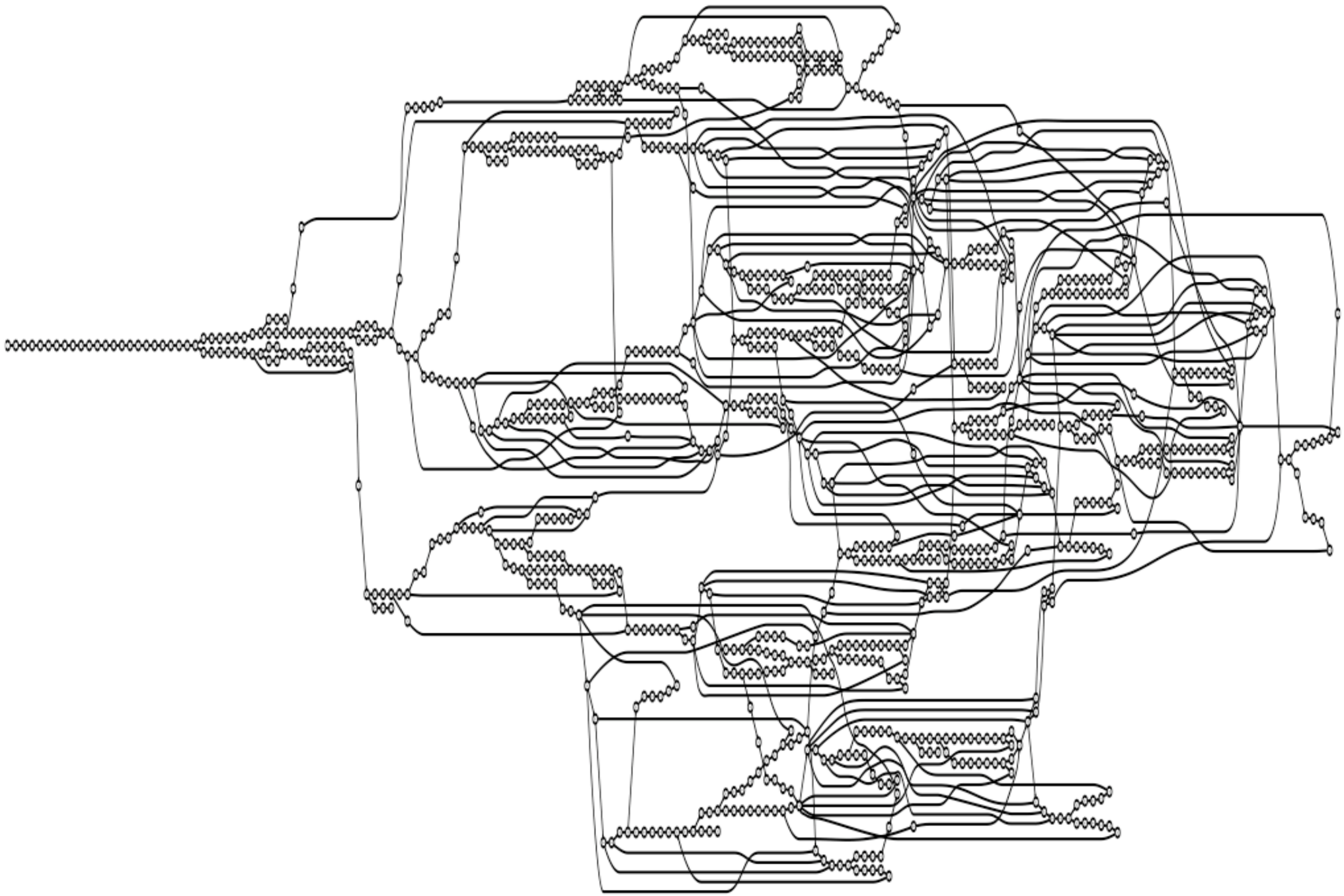

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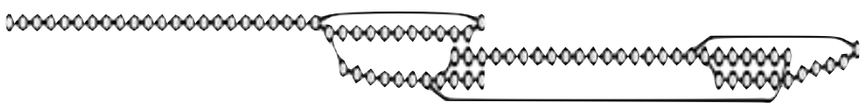
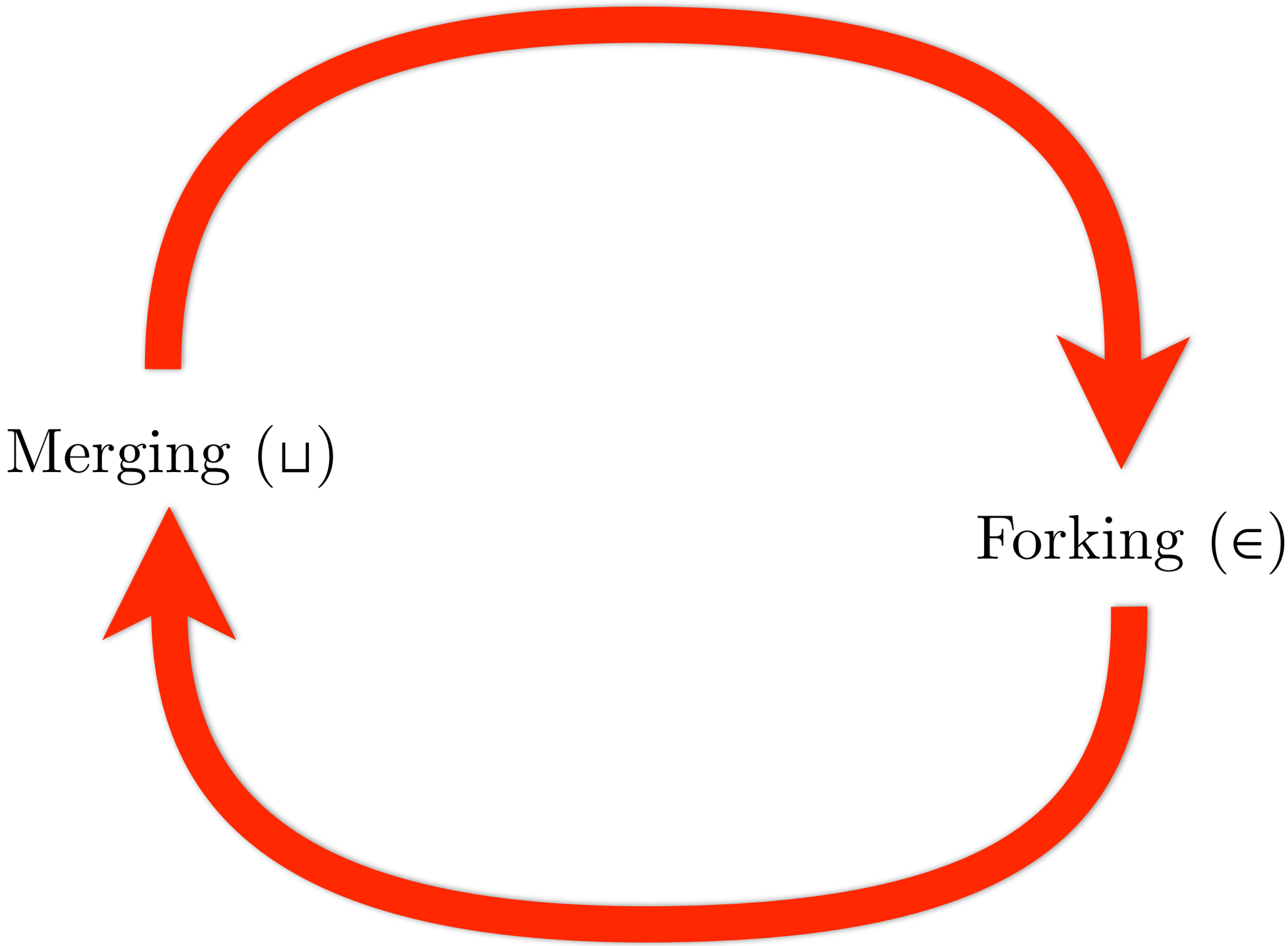
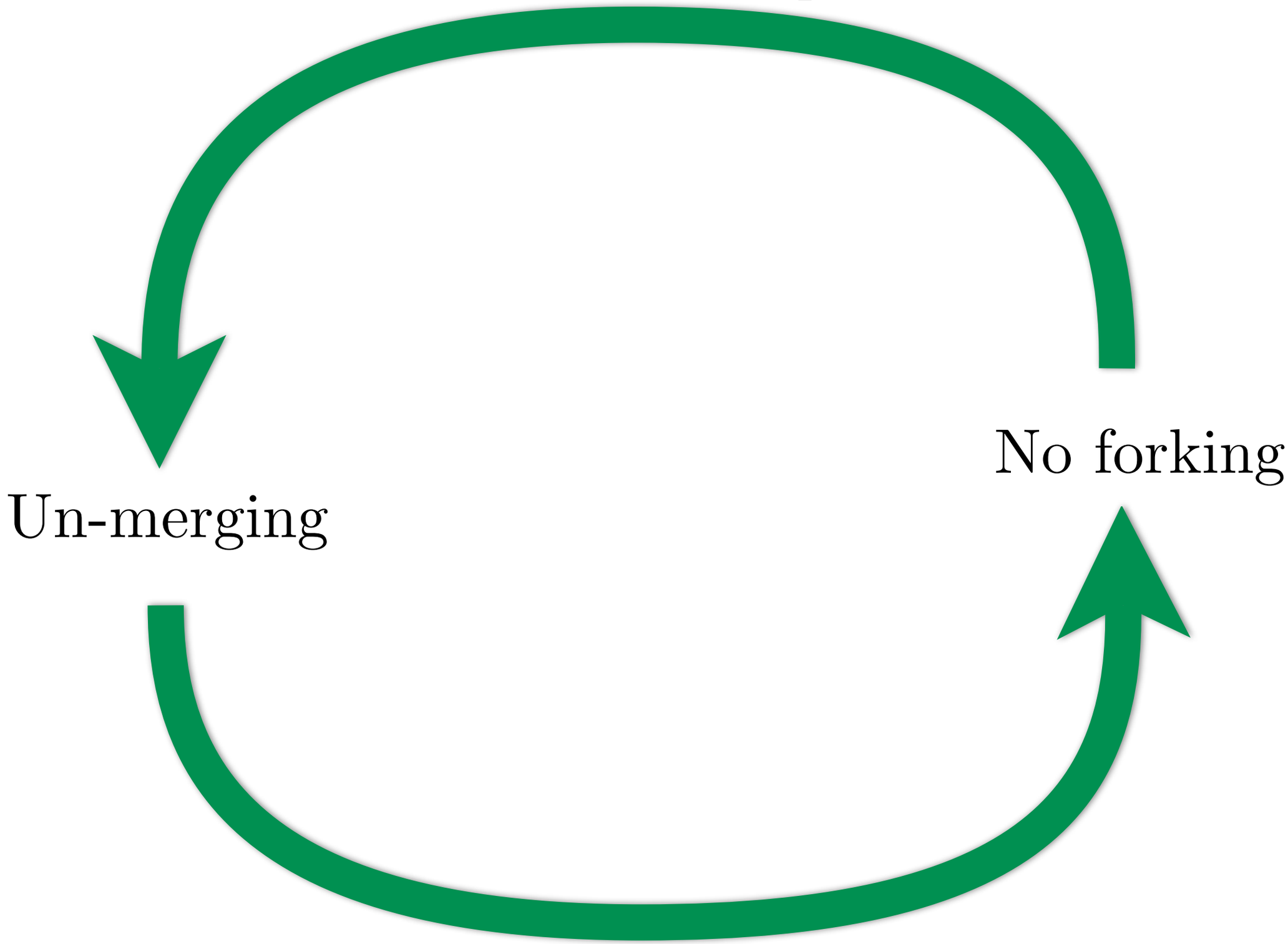


Vicious cyle



Virtuous cycle



Impact

- Orders-of-magnitude improvements

Implementing abstract garbage collection

The GC function

$$\hat{\Gamma} : \hat{\Sigma} \rightarrow \hat{\Sigma}$$

$$\hat{\Gamma}(\hat{\varsigma}) = (call, \hat{\beta}, \hat{\sigma} | \hat{\mathcal{R}}(\hat{\varsigma}), \hat{t})$$

where $\hat{\varsigma} = (call, \hat{\beta}, \hat{\sigma}, \hat{t})$

Reachable addresses

$$\hat{\mathcal{R}} : \hat{\Sigma} \to \mathcal{P}\left(\widehat{Addr}\right)$$

$$\hat{\mathcal{R}}(\hat{\varsigma}) = \left\{ \hat{a} : \hat{a}_0 \in Root(\hat{\varsigma}) \text{ and } \hat{a}_0 \overset{\hat{\sigma}_{\hat{\varsigma}}}{\rightsquigarrow} \hat{a} \right\}$$

Finding the root set

$$Root : \hat{\Sigma} \rightarrow \mathcal{P}\left(\widehat{Addr}\right)$$

$$Root(call, \hat{\beta}, \hat{\sigma}, \hat{t}) = range(\hat{\beta})$$

Reachability relation

Reflexive, transitive relation such that:

$$\frac{(lam, \hat{\beta}) \in \hat{\sigma}(\hat{a}) \qquad \hat{a}' \in range(\hat{\beta})}{\hat{a} \overset{\hat{\sigma}}{\rightsquigarrow} \hat{a}'}$$

Example: Without GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

Example: Without GC

```
let id = λx.x
         ---------
         id 3
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

Example: Without GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{x}) = \{3\}$

Example: Without GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{x}) = \{3\}$

$\hat{\sigma}(\mathtt{x}) = \{3, 4\}$

Example: With GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

Example: With GC

```
let id = λx.x
         ---------
         id 3
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

Example: With GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{a}) = \{3\}$

Example: With GC

```
let id = λx.x
    a  = id 3
    b  = id 4
in b
```

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{a}) = \{3\}$

$\hat{\sigma}(\mathtt{x}) = \{\}$

$\hat{\sigma}(\mathtt{b}) = \{4\}$

Applications

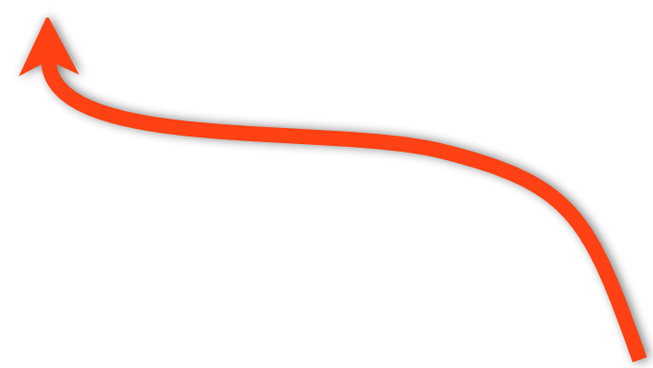
Inlining

Flow-based inlining

```
(f x)
```

Flow-based inlining

```
(f x)
```



```
(λ (z) z)
```

Flow-based inlining

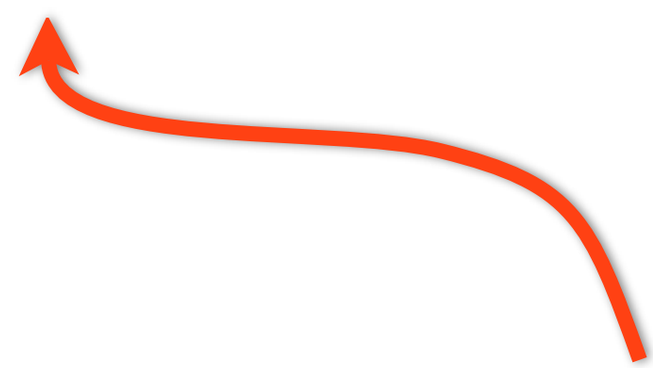
```
((λ (z) z) x)
```

Flow-based inlining

```
(f x)
```

Flow-based inlining

```
(f x)
```



```
(λ (z) y)
```

Flow-based inlining

```
((λ (z) y) x)
```

Flow-based inlining

Environment scenarios

Environment scenarios

- What if the free variables aren't in scope?

Environment scenarios

- What if the free variables aren't in scope?
- What if the free variables **are** in scope?

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
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Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
```

```
(λ () 3)
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
      (λ () 3)
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
      (λ () 3)
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      (h)
                      (λ () x)))))
  (f 0 (f 3 #f)))
```

```
(λ () 3)
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      ((λ () x))
                      (λ () x)))))
  (f 0 (f 3 #f)))
  (λ () 3)
```

Pathological example

```
(let ((f (λ (x h) (if (zero? x)
                      ((λ () 0))
                      (λ () x)))))
  (f 0 (f 3 #f)))
  (λ () 3)
```

Environment analysis

A little lesson

- My first solution: ΔCFA (Might & Shivers, 2006)
- Hallmarks of an elegant result, good dissertation
- Poor results in practice: explosions inevitable
- 2.5 years invested with ΔCFA -- a wrong turn

It’s never too late to stop making a bad decision.

</rant>

Environment problem

Environment problem

- Are two environments equivalent?

Environment problem

- Are two environments equivalent?
- What is **equivalent**?

Environment problem

- Are two environments equivalent?
- What is **equivalent**?
- **Which** two environments?

Equivalent environments

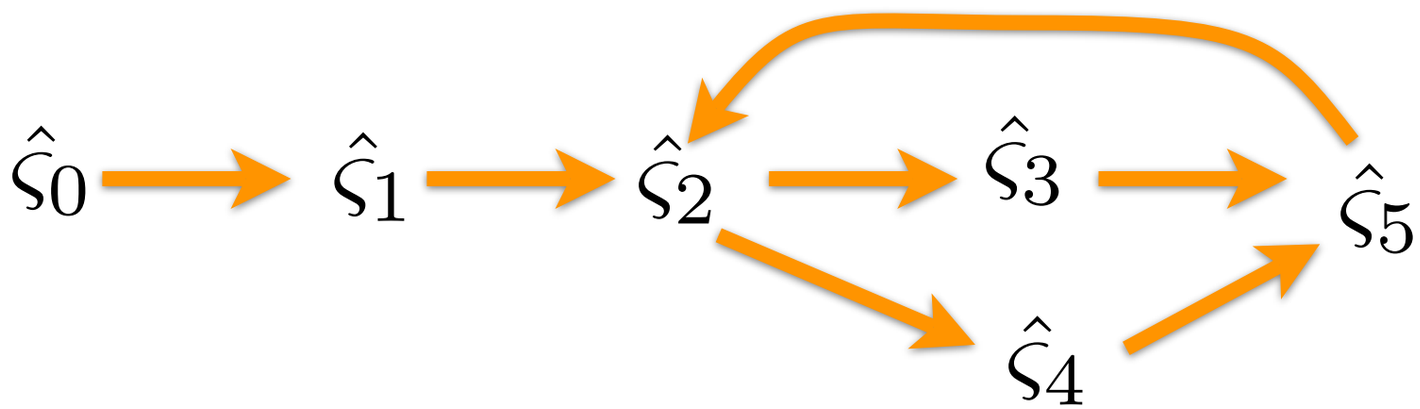
Environments ρ and ρ' are **equivalent**
over variables $v_1, \ldots, v_n$
if $\rho(v_i) = \rho'(v_i)$.

Equivalent environments

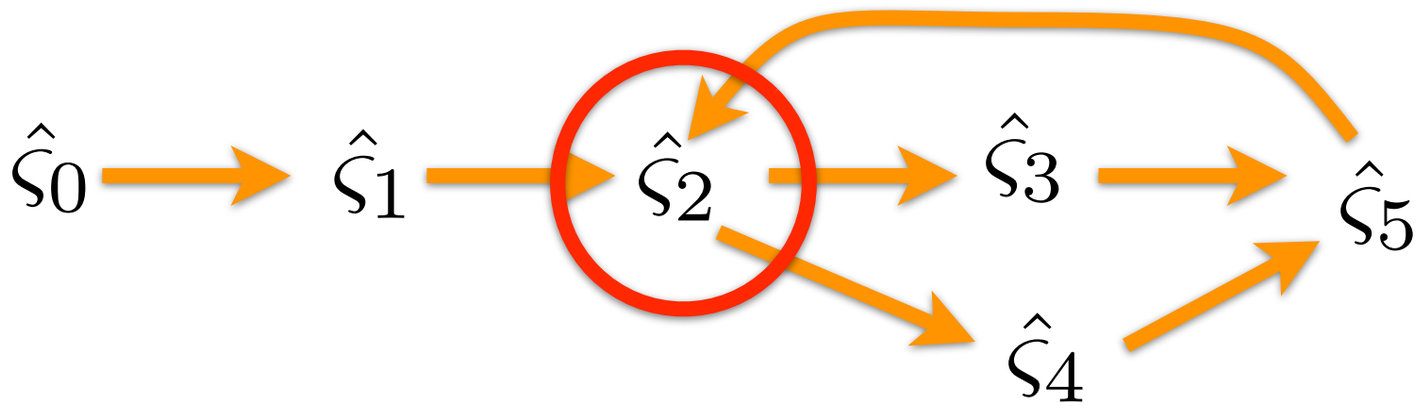
Environments β and β' are **equivalent** over variables $v_1, \ldots, v_n$ if $\beta(v_i) = \beta'(v_i)$.

Which environments

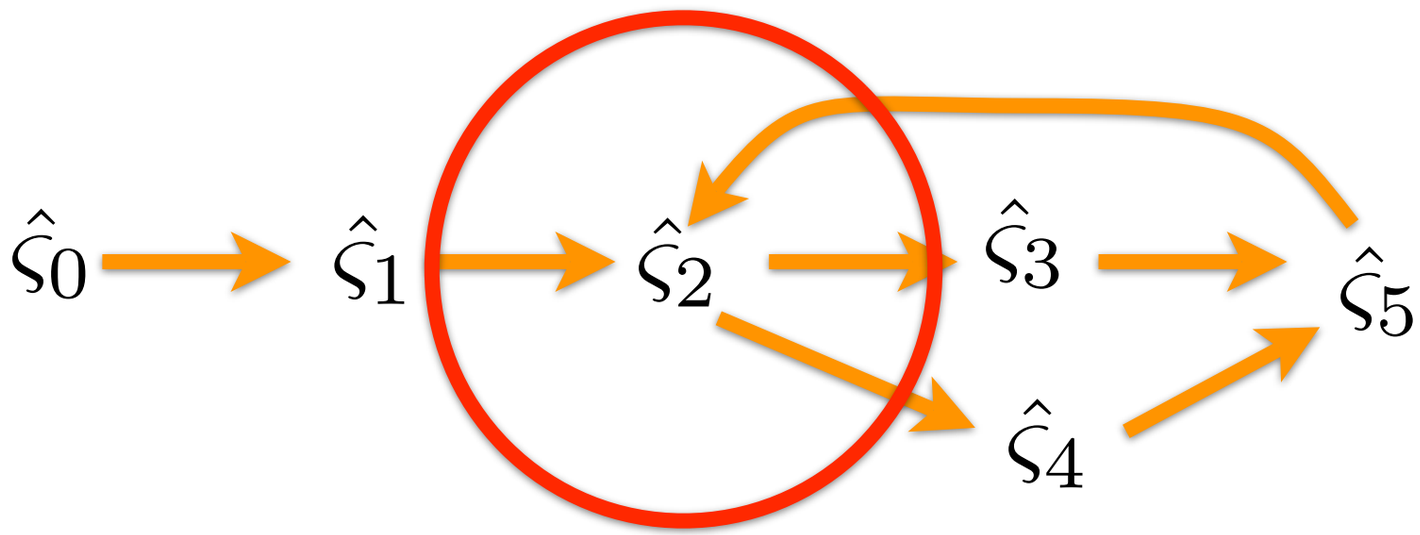
Which environments

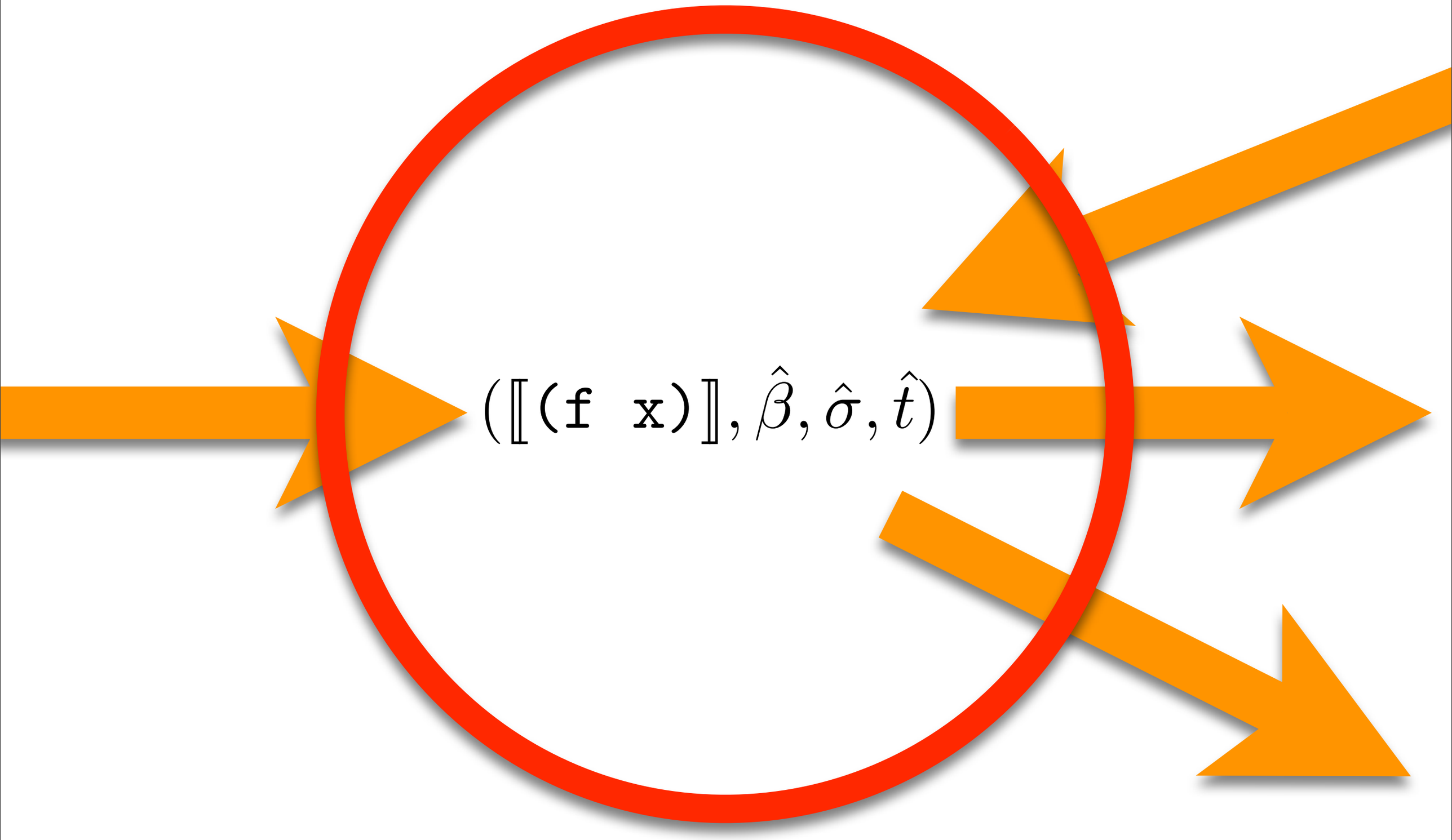


Which environments



Which environments




$([\![\texttt{(f x)}]\!], \hat{\beta}, \hat{\sigma}, \hat{t})$

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\xrightarrow{\texttt{f}}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{t})$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\uparrow \texttt{f}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{t})$$

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$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

$$\hat{\beta}'\llbracket \texttt{z} \rrbracket = \hat{a}$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$\hat{a} \equiv \{a_1\}$?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

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$\hat{a} \equiv \{a_1\}$?

$\hat{a} \equiv \{a_1, a_2\}$?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\uparrow \texttt{f}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

$$\hat{\beta}'\llbracket \texttt{z} \rrbracket = \hat{a}$$

The counting principle

If $\{x\} = \{y\}$, then $x = y$.

Abstract counting

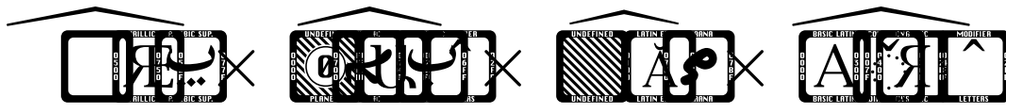
- Count abstract addresses as allocated
- When count is one, only one concrete

Counting machinery

$$\hat{c} \in \widehat{Count} = \{0, 1, \infty\}$$

$$\hat{\mu} \in \widehat{Max} = \widehat{Addr} \rightarrow \widehat{Count}$$

Abstract state-space

$\hat{\varsigma} \in \hat{\Sigma} = \mathsf{Call} \times$  $\times$ $\times$ $\times$

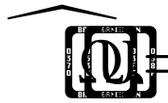
$\hat{\beta} \in$ $=$ $\mathsf{Var} \rightharpoonup$

$\hat{\sigma} \in$ $=$ $\rightarrow \hat{D}$

$\hat{d} \in \hat{D} = \mathcal{P}\left($ $\right)$

$=$ $=$

$=$ $=$

$=$  $=$ $\mathsf{Lam} \times$

$\hat{a} \in$ is a **finite** set of addresses

$\hat{t} \in$ is a **finite** set of time-stamps

Abstract state-space

$\hat{\varsigma} \in \hat{\Sigma} =$ Call $\times$ [illegible] $\times$ [illegible] $\times$ [illegible] $\times$ [illegible]

$\hat{\beta} \in$ [illegible] $=$ Var $\rightharpoonup$ [illegible]

$\hat{\sigma} \in$ [illegible] $=$ [illegible] $\rightarrow \hat{D}$

$\hat{d} \in \hat{D} = \mathcal{P}\left(\text{[illegible]}\right)$

[illegible] $=$ [illegible] $=$ [illegible]

[illegible] $=$ [illegible] $=$ [illegible]

[illegible] $=$ [illegible] $=$ Lam $\times$ [illegible]

$\hat{a} \in$ [illegible] is a **finite** set of addresses

$\hat{t} \in$ [illegible] is a **finite** set of time-stamps

Abstract state-space

When $call = [\![(f\ e_1 \ldots e_n)]\!]$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t}) \rightsquigarrow (call', \hat{\beta}'', \hat{\sigma}', \hat{\mu}', \hat{t}'), \text{ where}$$

$$(lam, \hat{\beta}') \in \hat{\mathcal{E}}(f, \hat{\beta}, \hat{\sigma})$$

$$\hat{C}_i = \hat{\mathcal{E}}(e_i, \hat{\beta}, \hat{\sigma})$$

$$lam = [\![(\lambda\ (v_1 \ldots v_n)\ call')]\!]$$

$$\hat{t}' = \widehat{tick}(call, \hat{t})$$

$$\hat{a}_i = \widehat{alloc}(v_i, \hat{t}')$$

$$\hat{\beta}'' = \hat{\beta}'[v_i \mapsto \hat{a}_i]$$

$$\hat{\sigma}' = \hat{\sigma} \sqcup [\hat{a}_i \mapsto \hat{C}_i]$$

$$\hat{\mu}' = \hat{\mu}' \oplus [\hat{a}_i \mapsto 1]$$

Abstract state-space

When $call = [\![(f\ e_1 \ldots e_n)]\!]$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t}) \rightsquigarrow (call', \hat{\beta}'', \hat{\sigma}', \hat{\mu}', \hat{t}'), \text{ where}$$

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$$lam = [\![(\lambda\ (v_1 \ldots v_n)\ call')]\!]$$

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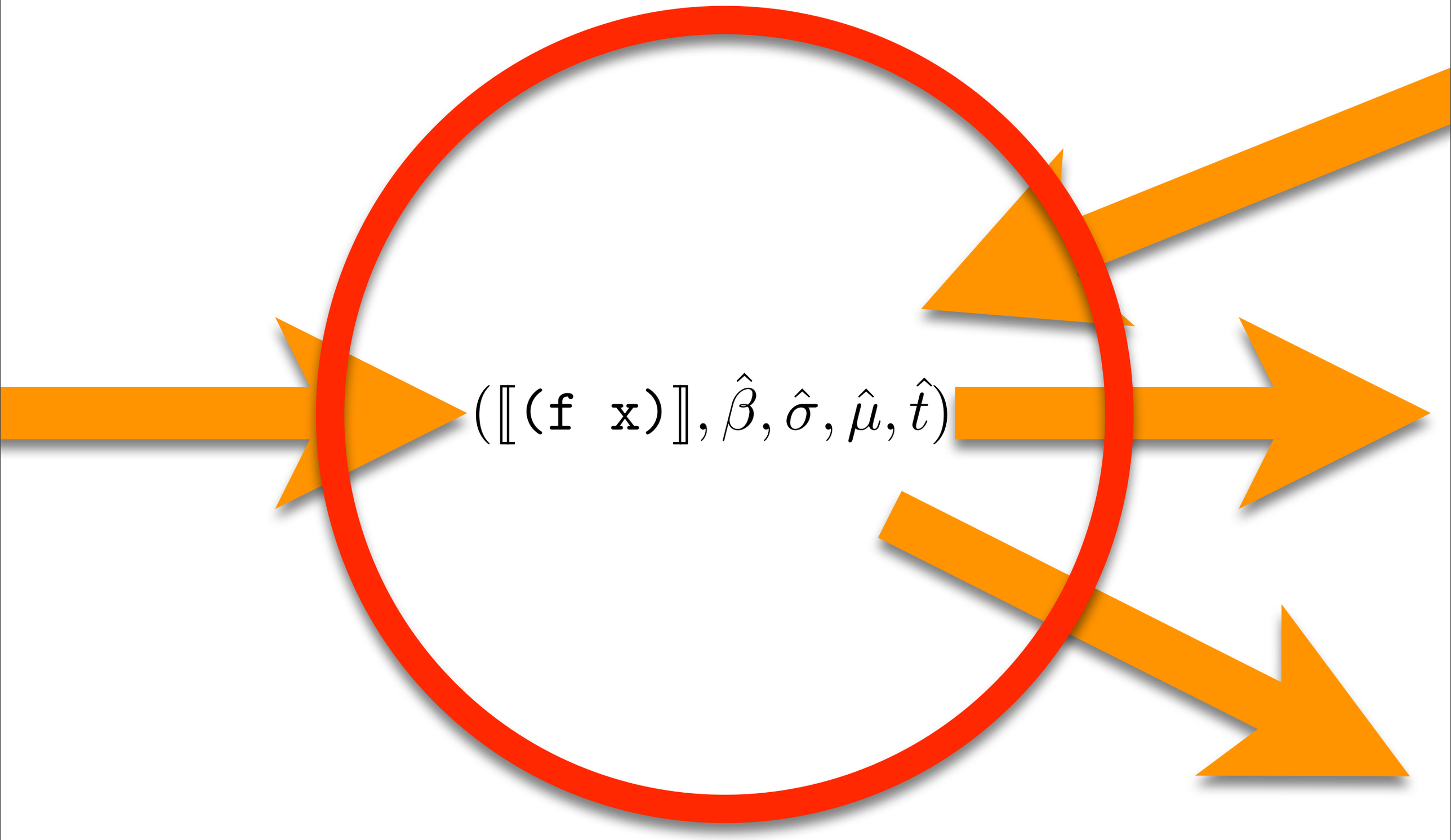
$$\hat{\mu}' = \hat{\mu}' \oplus [\hat{a}_i \mapsto 1]$$


$([\![\texttt{(f x)}]\!], \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\uparrow \texttt{f}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

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$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

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Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$\hat{a} \equiv \{a_1\}$?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\uparrow \texttt{f}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

$$\hat{\beta}'\llbracket \texttt{z} \rrbracket = \hat{a}$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$\hat{a} \equiv \{a_1\}$?

$\hat{a} \equiv \{a_1, a_2\}$?

$$(\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$$

$$\xrightarrow{\texttt{f}}$$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

$$\hat{\beta}'\llbracket \texttt{z} \rrbracket = \hat{a}$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$\hat{a} \equiv \{a_1\}?$ $\quad \hat{\mu}(\hat{a}) = 1$ $\quad ([\![(\lambda\ \ (\mathtt{x})\ \ \mathtt{z})]\!], \hat{\beta}')$

$\hat{a} \equiv \{a_1, a_2\}?$

$\mathtt{f}$

$$([\![(\mathtt{f}\ \ \mathtt{x})]\!], \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

$$\hat{\beta}[\![\mathtt{z}]\!] = \hat{a}$$

$$\hat{\beta}'[\![\mathtt{z}]\!] = \hat{a}$$

Are the counterparts to $\hat{\beta}$ and $\hat{\beta}'$ equivalent?

$\hat{a} \equiv \{a_1\}?$ $\quad \hat{\mu}(\hat{a}) = 1$ $\quad (\llbracket \texttt{(}\lambda\ \texttt{(x) z)} \rrbracket, \hat{\beta}')$

$\hat{a} \equiv \{a_1, a_2\}?$ $\quad \hat{\mu}(\hat{a}) = \infty$

$\texttt{f}$

$$(\llbracket \texttt{(f x)} \rrbracket, \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t})$$

$$\hat{\beta}\llbracket \texttt{z} \rrbracket = \hat{a}$$

$$\hat{\beta}'\llbracket \texttt{z} \rrbracket = \hat{a}$$

More applications of counting

CPS with `set!-then`

$$v \in \mathsf{Var}$$

$$f, e \in \mathsf{Exp} = \mathsf{Var} + \mathsf{Lam}$$

$$lam \in \mathsf{Lam} ::= (\lambda\ (v_1 \ldots v_n)\ call)$$

$$call \in \mathsf{Call} ::= (f\ e_1 \ldots e_n)$$

CPS with `set!-then`

$$
\begin{aligned}
v &\in \mathsf{Var} \\
f, e &\in \mathsf{Exp} = \mathsf{Var} + \mathsf{Lam} \\
lam &\in \mathsf{Lam} ::= (\lambda\ (v_1 \ldots v_n)\ call) \\
call &\in \mathsf{Call} ::= (f\ e_1 \ldots e_n) \\
&\qquad\quad \mid\ (\texttt{set!-then}\ v\ e\ call)
\end{aligned}
$$

Mutation w/o counting

When $call = [\![\texttt{(set!-then}\ v\ e\ call'\texttt{)}]\!]$:

$$(call, \beta, \sigma, t) \Rightarrow (call', \beta, \sigma', t'), \text{ where}$$
$$t' = tick(call, t)$$
$$d = \mathcal{E}(e, \beta, \sigma)$$
$$\sigma' = \sigma[\beta(v) \mapsto d]$$

When $call = [\![\texttt{(set!-then}\ v\ e\ call'\texttt{)}]\!]$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{t}) \rightsquigarrow (call', \hat{\beta}, \hat{\sigma}', \hat{t}'), \text{ where}$$
$$\hat{t}' = \widehat{tick}(call, \hat{t})$$
$$\hat{d} = \hat{\mathcal{E}}(e, \hat{\beta}, \hat{\sigma})$$
$$\hat{\sigma}' = \hat{\sigma} \sqcup [\hat{\beta}(v) \mapsto \hat{d}]$$

Strong update rule

When $call = [\![\texttt{(set!-then}\ v\ e\ call\texttt{)}]\!]$:

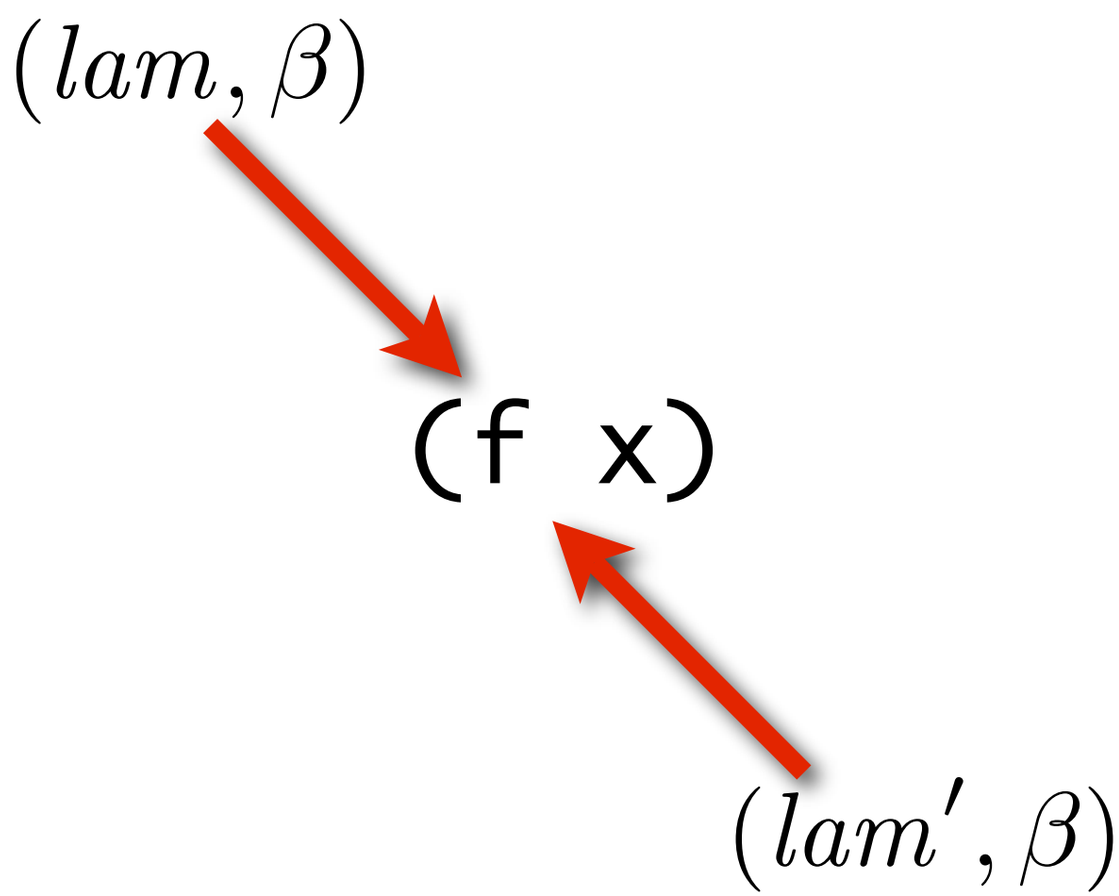
$([\![\texttt{(set!-then}\ v\ e\ call\texttt{)}]\!], \hat{\beta}, \hat{\sigma}, \hat{\mu}, \hat{t}) \rightsquigarrow (call, \hat{\beta}, \hat{\sigma}', \hat{\mu}, \hat{t}')$, where

$$\hat{t}' = \widehat{tick}(call, \hat{t})$$

$$\hat{d} = \hat{\mathcal{E}}(e, \hat{\beta}, \hat{\sigma})$$

$$\hat{\sigma}' = \begin{cases} \hat{\sigma}[\hat{\beta}(v) \mapsto \hat{d}] & \hat{\mu}(\hat{\beta}(v)) = 1 \\ \hat{\sigma} \sqcup [\hat{\beta}(v) \mapsto \hat{d}] & \text{otherwise.} \end{cases}$$

Lightweight closures



Globalization

- If a variable has max count of one,
- then it may be turned into a global.

Sparse interprocedural constant propagation

CPS with numerals and conditionals

$$v \in \mathsf{Var}$$

$$f, e \in \mathsf{Exp} = \mathsf{Var} + \mathsf{Lam}$$

$$lam \in \mathsf{Lam} ::= (\lambda\ (v_1 \ldots v_n)\ call)$$

$$call \in \mathsf{Call} ::= (f\ e_1 \ldots e_n)$$

CPS with numerals and conditionals

$$v \in \mathsf{Var}$$

$$f, e \in \mathsf{Exp} = \mathsf{Var} + \mathsf{Lam} + \mathsf{Num}$$

$$lam \in \mathsf{Lam} ::= (\lambda\ (v_1 \ldots v_n)\ call)$$

$$call \in \mathsf{Call} ::= (f\ e_1 \ldots e_n)$$

CPS with numerals and conditionals

$$
\begin{aligned}
v &\in \mathsf{Var} \\
f, e &\in \mathsf{Exp} = \mathsf{Var} + \mathsf{Lam} + \mathsf{Num} \\
lam &\in \mathsf{Lam} ::= \texttt{(}\lambda\ \texttt{(}v_1 \ldots v_n\texttt{)}\ call\texttt{)} \\
call &\in \mathsf{Call} ::= \texttt{(}f\ e_1 \ldots e_n\texttt{)} \\
&\qquad\quad \mid\ \texttt{(}\texttt{if}\ e\ call_1\ call_2\texttt{)}
\end{aligned}
$$

k-CFA with constants

$$\hat{\varsigma} \in \hat{\Sigma} = \mathsf{Call} \times \widehat{BEnv} \times \widehat{Store} \times \widehat{Time}$$

$$\hat{\beta} \in \widehat{BEnv} = \mathsf{Var} \rightharpoonup \widehat{Addr}$$

$$\hat{\sigma} \in \widehat{Store} = \widehat{Addr} \to \hat{D}$$

$$\hat{d} \in \hat{D} = \mathcal{P}\left(\widehat{Val}\right)$$

$$\widehat{val} \in \widehat{Val} = \widehat{Proc} + \widehat{Bas}$$

$$\widehat{proc} \in \widehat{Proc} = \widehat{Clo}$$

$$\widehat{clo} \in \widehat{Clo} = \mathsf{Lam} \times \widehat{BEnv}$$

$$\widehat{bas} \in \widehat{Bas} = \mathsf{Num}_{\top}^{\top}$$

$\hat{a} \in \widehat{Addr}$ is a **finite** set of addresses

$\hat{t} \in \widehat{Time}$ is a **finite** set of time-stamps

k-CFA with constants

$$\hat{\varsigma} \in \hat{\Sigma} = \mathsf{Call} \times \widehat{BEnv} \times \widehat{Store} \times \widehat{Time}$$

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$$\hat{\sigma} \in \widehat{Store} = \widehat{Addr} \to \hat{D}$$

$$\hat{d} \in \hat{D} = \mathcal{P}\left(\widehat{Val}\right)$$

$$\widehat{val} \in \widehat{Val} = \widehat{Proc} + \boxed{\widehat{Bas}}$$

$$\widehat{proc} \in \widehat{Proc} = \widehat{Clo}$$

$$\widehat{clo} \in \widehat{Clo} = \mathsf{Lam} \times \widehat{BEnv}$$

$$\boxed{\widehat{bas} \in \widehat{Bas} = \mathsf{Num}^{\top}_{\bot}}$$

$\hat{a} \in \widehat{Addr}$ is a **finite** set of addresses

$\hat{t} \in \widehat{Time}$ is a **finite** set of time-stamps

k-CFA with constants

When $call = [\![\texttt{(if}\ e\ call_t\ call_f\texttt{)}]\!]$, if $\{\texttt{0}\} \not\sqsubseteq \hat{\mathcal{E}}(e, \hat{\beta}, \hat{\sigma})$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{t}) \rightsquigarrow (call_t, \hat{\beta}, \hat{\sigma}, \hat{t}'), \text{ where}$$

$$\hat{t}' = \widehat{tick}(call, \hat{t})$$

k-CFA with constants

When $call = [\![\texttt{(if}\ e\ call_t\ call_f\texttt{)}]\!]$, if $\{\texttt{0}\} = \hat{\mathcal{E}}(e, \hat{\beta}, \hat{\sigma})$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{t}) \rightsquigarrow (call_f, \hat{\beta}, \hat{\sigma}, \hat{t}'), \text{ where}$$

$$\hat{t}' = \widehat{tick}(call, \hat{t})$$

k-CFA with constants

When $call = [\![\texttt{(if}\ e\ call_t\ call_f\texttt{)}]\!]$, if $\{\texttt{0}\} \sqsubset \hat{\mathcal{E}}(e, \hat{\beta}, \hat{\sigma})$:

$$(call, \hat{\beta}, \hat{\sigma}, \hat{t}) \rightsquigarrow (call', \hat{\beta}, \hat{\sigma}, \hat{t}'), \text{ where}$$

$$call' \in \{call_t, call_f\}$$

$$\hat{t}' = \widehat{tick}(call, \hat{t})$$

Wrap-up

- Techniques for 0CFA
- Frameworks for CFA
- Methods to improve
- Applications to drive

Questions?